From what is said above we may conclude not only that in the spontaneously pulsating heart there appear still other actions than those which we find expressed in the contraction, but also that these actions are to some extent independent. The actions not visible to the eye and characterized by definite electric phenomena, suggest the results of stimulation-processes as they can be shown also in the nerve without accompanying change of form. Though, however, the electrocardiogram may possess a certain independence of the form-cardiogram, the above communication does not in the least afford a reason to conversely come to the conclusion of the independence of the latter with respect to the former.

Mathematics. — "*On groups of polyhedra with diagonal planes, derived from polytopes*". By Prof. P. H. Schoute.

Introduction.

1. By "diagonal plane" of a polyhedron we understand any plane having only edges in common with the boundary of that body.[1])

There are two regular polyhedra admitting diagonal planes, the octahedron and the icosahedron. Through any edge of the octahedron passes one diagonal plane, containing the centre and bisecting the dihedral angle of the two faces passing through the edge. Through any edge of the icosahedron pass two diagonal planes; the angle formed by these planes and that formed by the two faces through the edge have the bisecting planes in common, and the cross-ratio between the couple of diagonal planes and the couple of faces has $\frac{1}{2}(3 - \sqrt{5})$ for one of its six mutually connected values.

The fact that only the two mentioned regular bodies possess diagonal planes is closely connected with this that through each of the vertices pass more than three faces. If we take away from the triangular faces meeting in a vertex the sides passing through that vertex, so as to retain of each the side opposite to this vertex, we find in the case of the octahedron a square adjacent to this vertex, in the case

[1]) In the last memoir of Dr. Fr. Schuh with the title "Over de meetkundige plaats, etc." (On the locus of the points in the plane, the sum of the distances of which to n given straight lines is constant, and analogous problems in space of three and more dimensions, Verhandelingen Kon. Akademie Amsterdam, first section volume IX, no. 5, 1908) occurs a series of polyhedra with the property that through any edge passes one diagonal plane. By extension to polydimensional spaces polytopes with diagonal spaces also make their appearance.

of the icosahedron a regular pentagon adjacent to this vertex, situated in a diagonal plane. Through any edge AB of the icosahedron pass two diagonal planes, as AB lies in two faces ABP and ABQ and therefore also in the diagonal planes corresponding to P, Q. Through any edge AB of the octahedron passes only one diagonal plane, as the third vertices P, Q of the faces ABP, ABQ through AB are opposite vertices and those points lead here to the same diagonal plane.

The diagonal planes of the icosahedron include a regular dodecahedron.

2. By "diagonal space" of a fourdimensional polytope we understand any space having only faces in common with the boundary of that polytope.

There are two regular cells admitting diagonal spaces, the C_{16} and the C_{600}. Through any face of the C_{16} passes one diagonal space, containing the centre and bisecting the dispatial angle of the two limiting bodies passing through the face. Through any face of the C_{600} pass two diagonal spaces; the angle formed by these spaces and that formed by the two limiting spaces through the face have the bisecting spaces in common, and the cross-ratio between the couple of diagonal spaces and the couple of limiting spaces has again — as we will prove afterwards — $\frac{1}{2}(3-\sqrt{5})$ for one of its six mutually connected values.

The fact that only the two mentioned regular cells possess diagonal spaces is again closely connected with this that through each of the vertices pass more than four limiting spaces and — we are obliged to add here — that these limiting spaces are tetrahedra [1]). If we take away from the limiting tetrahedra meeting in a vertex the faces passing through that vertex, so as to retain of each the face opposite to this vertex, we find in the case of the C_{16} a regular

[1]) This addition is necessary here. For the spatial sections of the regular C_{24} do not admit diagonal planes, though any vertex of this cell is situated in *six* of its limiting octahedra. As Mrs. A. Boole-Stott pointed out to me these spatial sections admit what we may call "would-be diagonal planes." If we consider — see fig. 64 of vol II of my "*Mehrdimensionale Geometrie*" — of the six octahedra meeting in A the squares adjacent to A, we get the six faces of a cube, the vertices and the edges of which are vertices and edges of C_{24}, whilst the faces of it are not faces of C_{24}. If C_{24} is cut by a space intersecting this cube, the vertices of the section which are points of intersection with edges of the cube will lie in a plane without all the sides of the polygon of intersection with these points as vertices being edges of the section. In the fourth part of my communication "On fourdimensional nets and their sections by spaces" I hope to be able to come back to this point.

octahedron adjacent to this vertex, in the case of the C_{600} a regular icosahedron adjacent to this vertex, situated in a diagonal space. Through any face ABC of the C_{600} pass two diagonal spaces, as ABC lies in two spaces $ABCP$, $ABCQ$ and therefore also in the diagonal spaces corresponding to P, Q. Through any face ABC of the C_{16} passes only one diagonal space, as the fourth vertices P, Q of the limiting spaces $ABCP$, $ABCQ$ through ABC are opposite vertices and these points lead here to the same diagonal space.

The diagonal spaces of the C_{600} include a regular C_{120}.

3. By "diagonal space Sp_{n-1}" of an n-dimensional polytope we understand any space Sp_{n-1} having with the boundary of this polytope only limiting spaces Sp_{n-2} in common.

Of the three regular polytopes, the simplex $S_{(n+1)}$ with $n+1$ vertices and $n+1$ limiting spaces Sp_{n-1}, the measure polytope M_n with 2^n vertices and $2n$ limiting spaces Sp_{n-1}, and the cross polytope Cr_n with reversely $2n$ vertices and 2^n limiting spaces Sp_{n-1}, only the last one possesses diagonal spaces Sp_{n-1}. Through any space Sp_{n-2} bearing a limiting simplex $S_{(n-1)}$ passes one diagonal space Sp_{n-1}, containing the centre and bisecting the angle between the two limiting spaces Sp_{n-1} passing through this Sp_{n-2}.

The fact that of the three regular polytopes only the cross polytope possesses diagonal spaces Sp_{n-1} is once more closely connected with this that through each of the vertices pass 2^{n-1} — and therefore more than n — limiting spaces Sp_{n-1}. If we take away from the limiting simplexes $S_{(n)}$ passing through any vertex the spaces Sp_{n-2} passing through this vertex, so as to retain the 2^{n-2} spaces Sp_{n-2} opposite to this vertex, we find the cross polytope Cr_{n-1} adjacent to this vertex, situated in a diagonal space Sp_{n-1}. Here too through any space Sp_{n-2} containing a limiting simplex $S_{(n-1)}$ pass two limiting spaces Sp_{n-1}. But, as the new vertices P and Q of the simplexes $S_{(n)}$ situated in these limiting spaces are opposite vertices of Cr_n leading to the same Cr_{n-1}, through each limiting simplex $S_{(n-1)}$ passes only one diagonal space Sp_{n-1}.

4. By intersecting a fourdimensional polytope, each face of which is situated in d diagonal spaces, by a space not containing an edge of the polytope, we get as section a polyhedron with the property that each of its edges is contained in d diagonal planes. For, if the intersecting space meets a face of the polytope, it meets also the d diagonal spaces passing through that face, and this always furnishes an edge of the section and d diagonal planes passing through it. So

the sections of the cells C_{16} and C_{600} by an arbitrarily chosen space are polyhedra with the property that through each edge passes respectively one diagonal plane or a couple of these. As four spaces passing in Sp_4 through the same face are cut by any space of Sp_4 in four planes through a line with the same cross-ratio, the sections of C_{600} by a space not containing an edge will be characterized by the property that the couples of faces and diagonal planes through an edge possess a constant cross-ratio. For from the regularity of C_{600} can be deduced that this cross-ratio is the same for all the faces, as we have stated already. Now the section of C_{600} by a space normal to an axis OE_0 (through a vertex E_0) is a regular icosahedron, if only the intersecting space is quite close to E_0 and this proves that the constant cross-ratio of C_{600} must be equal to that of the icosahedron.

5. Indeed, it is not difficult to show directly that the cross-ratio of C_{600} is really $\frac{1}{2}(3-\sqrt{5})$.

Let ABC be any face of C_{600} and O, P, Q (fig. 1) represent successively the centre of C_{600} and the fourth vertices of the two limiting tetrahedra $ABCP$, $ABCQ$ passing through ABC. Then the plane OPQ of the diagram will contain the centre of gravity G of the face ABC and be perfectly normal to this face in this point. From $GP=GQ$ and $OP=OQ$ can be deduced that the quadrangle $OPGQ$ is a deltoid with OG as axis of symmetry. As furthermore the normals GP' and GQ' dropped from G on OP and OQ are the traces of the plane of the diagram with the two diagonal spaces, we get for the cross-ratio $(PQRS)$

$$\frac{PR}{PS}:\frac{QR}{QS}=\left(\frac{PR}{PS}\right)^2=\left(\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}\right)^2=\frac{\sin^2(\alpha-\beta)}{\sin^2(\alpha+\beta)}.$$

Now if the edge of C_{600} is our unit and we represent for brevity's sake $\sqrt{5}$ by e we have (see my "*Mehrdimensionale Geometrie*", vol. II, p. 200)

$$OP=\frac{1}{2}(e+1),\ OP'=\frac{1}{4}(e+3),\ OG=\frac{1}{6}(e+3)\sqrt{3},\ PG=\frac{1}{3}\sqrt{6}.$$

From this ensues

$$\beta=60^\circ,\ \sin\alpha=\frac{1}{8}(e+1)\sqrt{6},\ \cos\alpha=\frac{1}{4}\sqrt{7-3e}$$

and therefore

$$(PQRS)=\left(\frac{e-1}{2}\right)^2=\frac{1}{2}(3-e)=0{,}381966\ldots$$ [1]

[1] In the same way the cross-ratio of the four planes through an edge of the icosahedron can be found.

6. In the third part of my communications "On fourdimensional nets and their sections by spaces", which is about to appear in these "Proceedings" we shall find occasion to fix attention on the diagonal planes presenting themselves in the sections of the C_{16}. As any vertex — or rather any couple of opposite vertices — of C_{16} possesses an adjacent octahedron, the polygons situated in these diagonal planes are always sections of octahedra. Probably the diagonal planes presenting themselves in the sections of the C_{600} were discovered for the first time by Mrs. A. BOOLE-STOTT, who made models of these sections, and explained as sections with diagonal spaces by Mr. H. W. CURJEL. [1])

The object of this paper is to study more closely the cases in which the intersecting space contains one or more edges of C_{16} and C_{600}; of the results revealed by these considerations these about C_{600} have especially roused our interest.

A. *The spatial sections through an edge of* C_{16}.

7. We consider the case in which the intersecting space contains the edge AB of C_{16} and indicate by A' and B' the vertices opposite to A and B. Then all the vertices except A and A' are adjacent to A and A', all the vertices except B and B' are adjacent to B and B', and so the four other vertices P_1, P_2, P_3, P_4 (fig. 2) are adjacent to A and B at the same time. In other words: the octahedra adjacent to A and B, situated in different spaces, penetrate one another in the square $P_1P_2P_3P_4$, the vertices of which they have in common. So through the edge AB pass two diagonal spaces, one of which corresponds to the opposite vertices P_1, P_3, the other to the opposite vertices P_2, P_4; they intersect the plane of the square $P_1P_2P_3P_4$, perfectly normal in O to the plane through AB and $A'B'$, respectively in the diagonals P_2P_4, P_1P_3. If l is the trace of the intersecting space through AB on the plane $P_1P_2P_3P_4$, and this line l, determining with AB that space, meets the diagonals P_2P_4, P_1P_3 in the points S_{13}, S_{24} situated within the square, then the section will show the particularity that the planes ABS_{13} and ABS_{24} are diagonal planes; so in some cases the edge AB will lie in *two* diagonal planes.

In the third communication "On fourdimensional nets, etc." quoted above will be indicated that the particularity of an edge being situated

[1]) A series of these models, showing e.g. the decomposition of the 120 vertices of the C_{600} into the vertices of five cells C_{24}, has been inserted lately into the collection of mathematical models of the University of Groningen.

in two diagonal planes does not present itself in the four groups of principal sections of C_{16}.

B. *The spatial sections through an edge of* C_{600}.

8. Through any edge AB (fig. 3) of C_{600} pass five limiting tetrahedra of this cell; the five edges opposite to AB of these tetrahedra are the sides of a regular pentagon $P_1P_2 \ldots P_5$, the vertices of which are at the same time adjacent to A and B. In other words: the icosahedra adjacent to A and B, situated in different spaces, penetrate one another in the regular pentagon $P_1P_2 \ldots P_5$, adjacent to AB, the vertices of which are common to both. So through the edge AB pass five diagonal spaces corresponding respectively to the five vertices $P_1, P_2, \ldots, P_5$; they intersect the plane of the pentagon, perfectly normal in its centre M to the plane ABM, in the diagonals $P_5P_2, P_1P_3, \ldots P_4P_1$ of the pentagon, or — if one likes — in the sides of the starpentagon $P_1P_3P_5P_2P_4$. In the case of C_{16} the centre O of the square $P_1P_2P_3P_4$ was at the same time the centre of the cell. Here the centre M of the pentagon is not even the centre of the two icosahedra penetrating one another, and still less the centre of C_{600}; here the line joining M to the midpoint M' of the edge AB must contain the centre O of C_{600}.

If the trace l of the intersecting space on the plane of the pentagon adjacent to AB cuts P_5P_2 in S_1 (fig. 3), ABS_1 is a diagonal plane. For this plane is the intersection of the intersecting space determined by AB and l with the diagonal space determined by AB and P_5P_2 of the icosahedron adjacent to P_1, and S_1 lies on P_5P_2 itself, not on its production. Indeed it is evident that this icosahedron is cut by any plane through AB and a point of P_5P_2, if this point lies on P_5P_2 itself, whilst the plane will contain of this icosahedron the edge AB only, if this point lies on P_5P_2 produced. In order to prove this we have only to observe that the lines AB and P_5P_2, the first of which is an edge of C_{600} and the latter a chord, cross one another normally. From this it ensues that these lines, likewise edge and chord of the icosahedron determined by the points A, B, P_5, P_2, can be represented (fig. 4), in projection on a plane through two opposite edges $pr, p'r'$ of the icosahedron, by the edge in q normal to the plane of the diagram and the chord pp' situated in that plane, the extremities of the edge being joined by edges to the extremities p, p' of that chord. This shows immediately that any plane through the edge projecting itself in q cuts the icosahedron or not, according to whether the point of intersection of the plane with pp' lies on this line itself or on its production.

So for the position of the intersecting space adopted in fig. 3 four diagonal planes ABS_1, ABS_2, ABS_3, ABS_4 pass through AB; the point of intersection S_5 of l and P_4P_1 falls on the production of this side and does not lead to a diagonal plane.

On each side P_5P_2 of the starpentagon (fig. 3) there are remarkable points besides the extremities P_5, P_2, which lead to faces and not to diagonal planes, namely the points of intersection Q_3, Q_1 with the other sides and the midpoint M_1. If S_1 coincides with Q_3, the sides P_5P_2 and P_4P_1 are cut in the same point and, the two corresponding diagonal planes coinciding with one another in the plane of intersection of the diagonal spaces ABP_5P_2 and ABP_4P_1 adjacent to P_1 and P_5, this plane must contain the pentagon adjacent to the edge P_1P_5 of C_{600}. So in this case the polygon situated in the diagonal plane — compare in fig. 4 the planes normal to the plane of the diagram in the lines qr and qr' — is a regular pentagon. If S_1 coincides with M_1 the plane ABM — compare fig. 4 —, being a plane of symmetry of the icosahedron, contains AB and the edge parallel to AB.

9. My second memoir with the title "Regelmassige Schnitte u.s.w." Regular sections and projections of C_{120} and C_{600}, Verhandelingen Amsterdam, first section, vol. IX, N°. 4, 1907 contains the data that enable us to determine, for any position of the intersecting space containing a certain number of edges of C_{600} belonging to the four groups of sections studied there, the number and the position of the diagonal planes passing through any one of these edges, and to construct the icosahedral sections situated in these planes. We will try to explain this shortly.

On the righthand side of the plates II, IV, VI, VIII has been indicated how the icosahedra adjacent to the vertices of C_{600} project themselves on the axes OR_0, OF_0, OK_0, OE_0. In order to see at a glance which sections normal to these axes do contain edges of icosahedra — and therefore also of C_{600} — we consult the upper lines of the plates XVIII, XVI, XIV, XII. We find then the following table:

II^b, XVIII	$a_1(6)$,	$d_1(12)$,	$e_1(12)$,	$f_1(6)$,		
IV^b, XVI	$a_2(3)$,	$c_2(3)$,	$e_2(3)$,	$f_2(6)$,	$h_2(6)$,	$i_2(3)$,
VI^b, XIV	$a_3(1)$,	$b_3(5)$,	$c_3(10)$,	$f_3(5)$,	$g_3(10)$,	$i_3(10)$,
VIII^b, XII	$b_4(30)$,	$c_4(30)$,	$e_4(60)$,			

in which the indices 1, 2, 3, 4, distinguishing the groups, correspond to those of the groups of icosahedra on (II^b), IV^b, VI^b, VIII^b, whilst the numbers placed between brackets indicate how many edges lie in the intersecting spaces. However the cases a_1, a_2, a_3 can be left out, as referring to intersecting spaces leaving the C_{600} totally on one side and being therefore unable to furnish sections containing diagonal planes; for each of the sixteen remaining cases the trace l of the intersecting space on the plane of the pentagon adjacent to the chosen edge must be constructed. These traces, indicated by the symbols $d_1, e_1, \ldots, e_4$ of the cases to which they belong, are represented altogether in fig. 5.

10. The determination of the trace l causes the least trouble if this line contains two of the remarkable points P_i, Q_i, M_i corresponding respectively to a vertex, a point of intersection of two non-adjacent sides and the midpoint of a side of the starpentagon. In order to divide the difficulties we treat these simple cases first.

Case d_1. On plate II^b we find under d that the groups I and VII, each containing four icosahedra, furnish faces situated in the intersecting space, whilst group III gives six icosahedral sections through two opposite edges. So the trace d_1 to be found passes through a vertex P_i and a midpoint M_i; if P_1 is taken as P_i, then M_i must be either M_3 or M_4. So we find that the trace d_1 coincides with one out of ten homologous lines, if by "homologous" lines we mean lines passing into one another either by a rotation of the pentagon about its centre M to an amount of any multiple of 72° or by a reflexion with respect to one of the lines MP_i as mirror, i.e. in general by any transformation that transforms the pentagon into itself.

The line d_1 cuts two other sides, the sides P_5P_2 and P_3P_5, of the starpentagon; as P_1 does not lead to a diagonal plane, any of the 12 edges lying in the intersecting space is contained in three diagonal planes. These new diagonal planes are connected with the groups IV and VI, each of which contains 12 icosahedra. As the section passes rather near the centre M_i in the case of IV and rather near one of the extremities P_i in the case of VI, it is probable that IV corresponds to the point on P_5P_2, VI to the point on P_3P_5. Later on we will prove this to be true.

We will add the remark, that the number 12 of the edges lying in the intersecting space is given back by each of the groups I, III, IV, VI, VII, the corresponding diagonal planes — the *faces* of I and VII included — containing successively 3, 2, 1, 1, 3 edges.

Case c_2. On plate VI^b under c the group I_8 leads to a point Q

and the group II_3 to a point M_i; if Q_1 is chosen as Q_i, M_i must coincide either with M_2 or with M_5.

The chosen line c_3 furnishes one point of intersection more, on P_5P_2; so there must be one group of icosahedra more with an edge situated in the intersecting space. Indeed, we find only one group VI_3, IV_8 belonging again to Q_1.

C a s e e_4. On plate VIII^b under e we have to deal with a central section of C_{600}, from which ensues that the line e_4 passes through the centre M of the pentagon. Moreover the groups III_4, IV_4, V_4 furnish successively a point P_i, a point Q_i, a point M_i. So e_4 is a diameter through a vertex of the pentagon, e.g. $P_1M_1Q_1$. Here no other point of intersection appears.

11. It would be possible to go on in this manner and to treat successively, proceeding from the easier cases to the more difficult ones, the remaining lines through two remarkable points, the lines through only one remarkable point, the lines parallel to one of the sides. We prefer however to explain now, for an arbitrary case, how the ratio of division of the side of the starpentagon corresponding to a determined group of icosahedra can be found by means of Fig. 3 of the quoted memoir, which is repeated here with slight modification as fig. 4.

We therefore consider the group IV_1 of plate II^b mentioned above under d_1, and remember that the icosahedral sections corresponding to this group are determined, according to the quoted memoir, by planes normal to the plane of fig. 4 in a line parallel to pp^I. If the edge normal in q to the plane of that diagram is once more the edge AB and the chord pp' situated in that plane the side of the starpentagon, then the point S on that side determining the diagonal plane in question is found by drawing through q the line qS parallel to pp^I. Now pw is the smaller segment of the line pp' divided internally in medial section and the same relation holds for $p^{II}r = wq$ with respect to the segments $p^Ir = p^{II}s = sq$. So if the ratio of the side of the regular pentagon to its diagonal is indicated by $\frac{s}{d}$, we deduce from similar triangles

$$p^Iw : wq = pw : wS,$$

which may be transformed into

$$p^Iq : p^Iw = pS : pw\,.$$

This leads to

$$\frac{pS}{pp'} = \frac{3d+2s}{3d+s} \cdot \frac{pw}{pp'} = \frac{3d+2s}{3d+s} \; \frac{d-s}{d} = \frac{(2+e)(3-e)}{5+e} = \frac{1+e}{e+5} = \frac{1}{5}\,e.$$

19*

By this value the place of S on pp' is perfectly determined; however in fig. 5 we may — and, if d_1 has been determined by P_1 and M_1, we must — assume for S not the point on the right of M_1 corresponding to this ratio but the point on the left.

As a second example we consider the group IX$_2$ of plate VI^b to which — according to the second memoir — corresponds a series of planes normal to the plane of fig. 4 parallel to up_2^{III} (p_2^{III} being the midpoint of sv). We draw through s and q the lines sS'' and qS' parallel to up_2^{III} and determine now the ratio of pS' to pp' by means of similar triangles as follows. These triangles give

$$\frac{S'w'}{qw'}=\frac{w'S''}{w'w''}=\frac{pS''}{ps}=\frac{pS''}{w'u}=\frac{pw'}{w''u}=\frac{pw'}{\frac{1}{2}qu}.$$

So we have

$$\frac{S'w'}{pp'}=\frac{2qw'}{qu}\cdot\frac{pw'}{pp'}=\frac{2s}{d+s}\cdot\frac{s}{d}$$

and finally

$$\frac{pS'}{pp'}=\frac{pw'-S'w'}{pp'}=\frac{s}{d}\left(1-\frac{2s}{d+s}\right)=\frac{s\,(d-s)}{d\,(d+s)}=\frac{(e-1)\,(3-e)}{2\,(e+1)}=\tfrac{1}{2}\,(7-3e)\,.$$

In this way is obtained the complete system of the twelve different ratios λ given in the following table, where, when λ differs from $\frac{1}{2}$, the value smaller than $\frac{1}{2}$ always appears. For all the groups in any horizontal row λ has the value indicated in the last column but one. In the last column are given the numbers of centimeters corresponding to these ratios, when the length of the side of the starpentagon (fig. 5) is 20 centimeters. Finally the last column but two indicates the direction of the trace of the intersecting planes normal to the plane of fig. 4, by means of which the values of λ have been calculated. (See table p. 287).

For the sake of clearness the values of λ with the side (20 centimeters) of the starpentagon of fig. 5 as unit have been indicated separately in fig. 6. By transferring this scale division in fig. 5 to each of the sides P_6P_2, etc. we are enabled to draw immediately each of the traces l in question with accuracy.

12. By means of the preceding the polygon of intersection of the polyhedron situated in any assigned diagonal plane can be constructed. To this end we indicate in fig. 7, which is a repetition of fig. 4, for the twelve different cases of the table by the numbers 1, 2, . . . , 12 the traces of the intersecting planes passing through the edge in q normal to the plane of the diagram, and show how we can obtain all the measures necessary for the construction of these

Nr.	Groups					λ	
1	I_1, II_1, VII_1	II_2, $\overline{XI_2}$		III_4	qp	0	0
2	III_1	I_2, VII_2	II_3, V_3	V_4	qO	$\frac{1}{2}$	10
3			I_3, IV_3, $\overline{IX_3}$	II_4, IV_4	qs	$\frac{1}{2}(3-e)$	7.63932
4	IV_1		$VIII_3$		pp^I	$\frac{1}{5}e$	8 94427
5	V_1	IV_2			pp^{II}	$2(e-2)$	9.44272
6	VI_1		III_3		pp^{III}	$e-2$	4.72136
7	$VIII_1$	V_2			pp^{IV}	$\frac{1}{4}(e-1)$	6.18034
8		VI_2			pp_2^{II}	$\frac{1}{6}(5-e)$	9 14207
9		IX_2			up_2^{III}	$\frac{1}{2}(7-3e)$	2.91796
10		X_2	VI_3		qp_2^{IV}	$\frac{1}{10}(5-e)$	5.52786
11		XI_2			Op^{IV}	$\frac{1}{4}(3e-5)$	8.54102
12			IX_3		Or	$\frac{1}{4}(3-e)$	3.81966

polygons represented in fig. 8 by laying down in the plane of the diagram of fig. 7 the regular pentagon projecting itself in psv and the equilateral triangle projecting itself in rv. By the remark that all these polygons admit an axis of symmetry, the line k bisecting the edge $q_1 q_2$ normally, and that the measures qab, aa' of the pentagon of Nr. 9 and $qcde$, dd', ee' of the octogon of Nr. 4 used in fig. 8 are taken from fig. 7 this construction will become sufficiently clear. [1])

We add to this the following simple general remark. The polygon situated in a diagonal plane of which one of the sides is an edge of C_{600} is always either a pentagon, or a hexagon, or an octagon. If we once more determine the diagonal plane by means of the edge normal in q to the plane of fig. 4 and the point of intersection S with pp', then the section is a pentagon if S lies between p and

[1]) The letters a and c, that had to indicate points on k, have been omitted in fig. 8.

w or between w and p' and an octagon if S lies between w and w', except when S coincides with the midpoint in which case the section is a hexagon. In other words, with reference to the side $P_5 P_2$ of the starpentagon of fig. 5: the section is a hexagon if S coincides with M_1, an octagon if S lies elsewhere between Q_3 and Q_4, a pentagon if S falls between P_5 and Q_3 or Q_4 and P_2. So in the case h_2 we find two pentagons, since two points of intersection lie outside the pentagon with the vertices Q_i, a hexagon and an octogon, etc.

13. The method developed here has a slight drawback, revealing itself to the utmost in the determination of the exact position of the trace h_2. The difficulty consists in this that the method leaves us in the dark as to the succession of the different values of λ on the trace l. If we have deduced that the different ratios of VI_2, VII_2, IX_2, X_2 present themselves and we have chosen for VII_2 the centre M_3 (fig. 5) we are obliged to investigate by a rotation of the ruler about M_3 on which side — and in which of the two different points on this side — we must assume the point of division corresponding to VI_2 in order to make the other points of intersection to correspond to IX_2 and X_2. We now indicate finally how this difficulty can be overcome.

To any chosen edge of C_{600} projecting itself on plate IV^b in h on the axis OF_0, there correspond five adjacent points of C_{600}. If now it were possible:

1. to select a determined edge projecting itself in h on OF_0,

2. to point out the five adjacent vertices and to indicate in what order these points are the vertices of a regular starpentagon,

3. to find where these five points project themselves on the same axis OF_0,

then it would be possible to make out, in what ratio the successive sides of the starpentagon were divided in projection on OF_0 by h, which would enable us to fix in fig. 5 on each of these sides a quite definite point. Really in these suppositions the difficulty would be quite dissolved.

Now these suppositions are quite realisable, by means of the tables published in my first memoir with the title "Regelmässige Schnitte u. s. w." (Regular sections and projections of C_{120} and C_{600}, Verhandelingen Amsterdam, first section, vol. II, No. 7, 1894); we will explain this with the aid of fig. 9 for the case of the trace h_2.

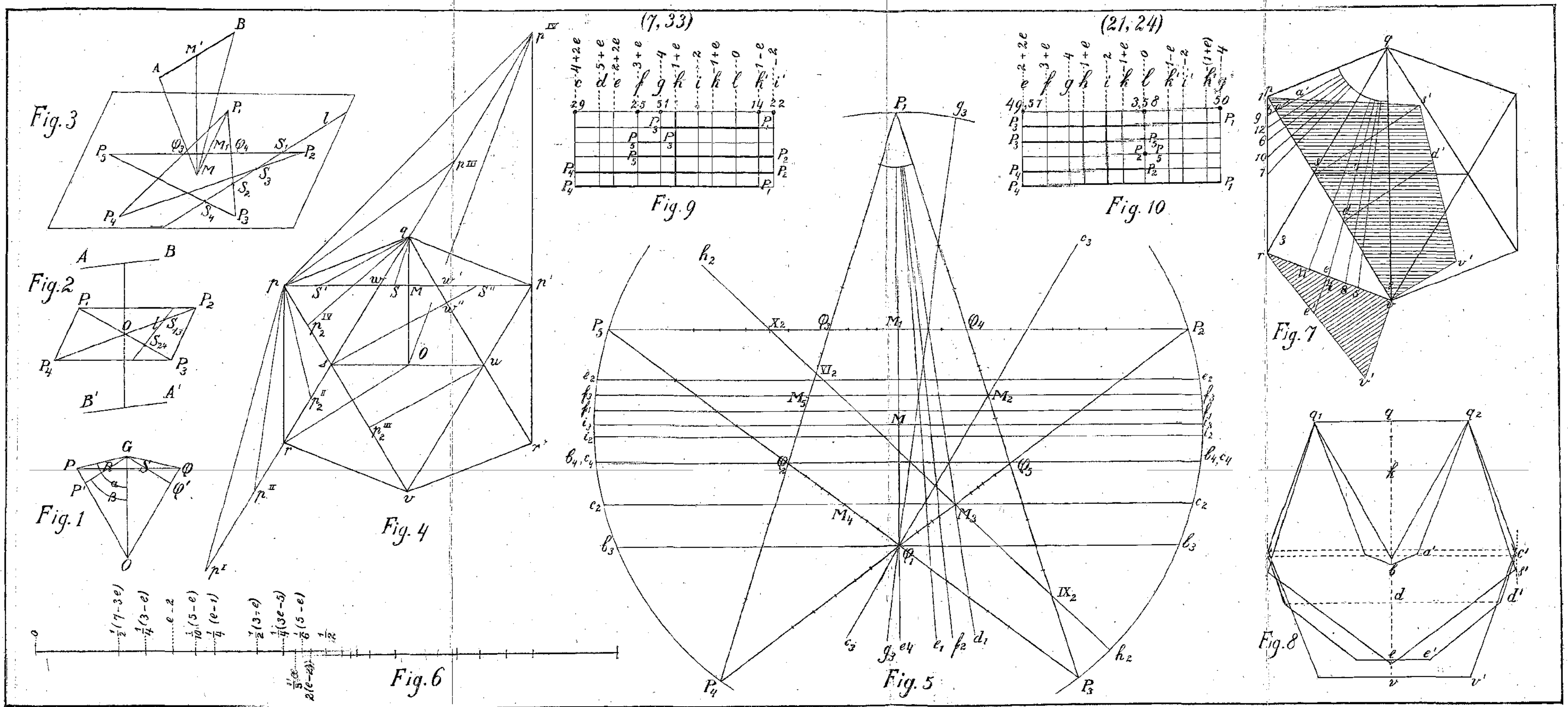
Fig. 3
Fig. 2
Fig. 1
Fig. 4
Fig. 6
(7, 33)
Fig. 9
(21, 24)
Fig. 10
Fig. 5
Fig. 7
Fig. 8

14. In "Tabelle I" with the inscription "Coordinatenstellung des Z^{600}" we find, if under "C, Zweite Querlinie" the z_1 corresponds to the chosen axis OF_0, that the vertices

$$-6,\ 7,\ -11,\ -12,\ 17,\ -18,\ 19,\ 20,\ 33,\ -34,\ 35,\ 36$$

have $1+e$ for value of z_1 and project themselves therefore in h — compare plate IV^b of the second memoir. From "Tabelle II" with the inscription "Kanten des Z^{600}" we then deduce that (7,33) is an edge of C^{600}, that 14, 22, 25, 29, 51 are the five vertices adjacent to this edge (7,33) and these points form a regular pentagon $P_1P_2P_3P_4P_5$ in the order of succession 14, 22, 51, 29, 25 and therefore a regular starpentagon $P_1P_3P_5P_2P_4$ in the order 14, 51, 25, 22, 29. Turning back to the column z_1 of "Tabelle I" we find at last that these vertices 14, 51, 25, 22, 29 admit successively for z_1 the values

$$1-e,\ 4,\ 3+e,\ -2,\ 2\,(2+e),$$

from which ensues that they project themselves — compare again plate IV^b of the second memoir — in k', g, f, i', c. This result is indicated in fig. 9. While the segments of the horizontal lines appearing there from right to left are indicated as to their relative length by

$$d,\ s,\ d,\ d,\ s,\ d,\ s,\ d,\ d,\ s,$$

we find, if we indicate by S the point on any side of the starpentagon projecting itself in h,

$$\frac{P_3S}{P_3P_1}=\frac{s}{3d+2s}=\frac{1}{2}(7-3e),\qquad \frac{P_5S}{P_5P_2}=\frac{d+s}{4d+3s}=\frac{1}{10}(5-e),$$

$$\frac{P_4S}{P_4P_2}=\frac{1}{2}\qquad,\qquad \frac{P_1S}{P_1P_4}=\frac{3d+s}{6d+3s}=\frac{1}{6}(5-e).$$

These results are in accordance with what has been found before; moreover they indicate quite definitely the place of each point of division [1]).

15. If we apply the new method to the case of a trace as e_2 parallel to one of the sides of the starpentagon, then the point S projecting itself in e on plate IV^b will have to divide the side P_5P_2 externally into the ratio unity and this requires, as S does not lie at infinity, that the edge P_5P_2 projects itself on OF_0 as a point.

[1]) As the second method gives something more in one respect than the first, it might seem superfluous to communicate the first. We are not of this opinion. For the first method has this advantage above the second that it leads immediately to a construction of the polygon situated in the diagonal plane as the section of a definite icosahedron by a definite plane.

This case is represented in fig. 10 for the edge (21, 24), where 3, 49, 50, 57, 58 are the five adjacent points, whilst 50, 3, 49, 57, 58 appears as the pentagon $P_1P_2P_3P_4P_6$, 50, 49, 58, 3, 57 as the starpentagon $P_1P_3P_5P_2P_4$. Really P_5P_2 projects itself into a point; moreover P_3P_1 and P_4P_1 on one hand and P_4P_2 and P_3P_6 on the other coincide in projection, which is closely connected with this that λ is the same for the two constituents of each pair.

Mathematics. — *"On triple systems, particularly those of thirteen elements."* By Dr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of September 26, 1908).

In a paper to this Academy[1]) Prof. J. DE VRIES gave a triple system of 13 elements of a different type than the cyclic system of Prof. NETTO[2]); he added however the observation, that no proof has been furnished of these types being the only ones.

Mr. K. ZULAUF shows in his dissertation[3]) that the systems given formerly by KIRKMAN (1853) and REISZ (1859) are identical to that of DE VRIES, so that the number of *known* systems is *two*; neither is anything here decided about the number of *possible* systems.

It seemed desirable to decide upon this point by means of a special investigation[4]). To this end some facility is offered by using those expressions which are used in the theory of the configurations, by regarding the 13 elements as *points*, the 26 triplets as *lines* which bear three of the points; the whole of the triple system then becomes the scheme of a diagonalless Cf. $(13_6, 26_3)$ where it is irrelevant whether this Cf. can be geometrically realized or not. A classification of these Cff. is now our aim in view.

The rest figure of the second order of a line of such a Cf., i.e. what remains if we leave out that line with its three points and the 3×5 lines passing through these points, is of necessity a Cf. (10_3), the 10 points of which are in three ways perspective and that according to the three points left out.

But then reversely each imaginable Cf. $(13_6, 26_3)$ of the desired type is obtained by

1st. starting from all possible Cff. (10_3),

2nd. by constructing for each Cf. (10_3) the Cf. $(10_3, 15_2)$ of its diagonals,

1) *Versl. Kon. Akad. v. Wet.* III, p. 64, 1894.

2) *Substitutionentheorie*, p. 220; *Math. Annalen*, Vol. 42.

3) *"Ueber Tripelsysteme von 13 Elementen"*, Giessen, 1897.

4) I subsequently find this question treated also by DE PASQUALE (*Rendic. R. Ist. Lombardo*, 2nd Ser., 32, 1899).