Citation:

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This case is represented in fig. 10 for the edge (21, 24), where 3, 49, 50, 57, 58 are the five adjacent points, whilst 50, 3, 49, 57, 58 appears as the pentagon $P_1P_2P_3P_4P_6$, 50, 49, 58, 3, 57 as the starpentagon $P_1P_3P_5P_2P_4$. Really P_5P_2 projects itself into a point; moreover P_3P_1 and P_4P_1 on one hand and P_4P_2 and P_3P_6 on the other coincide in projection, which is closely connected with this that λ is the same for the two constituents of each pair.

Mathematics. — *"On triple systems, particularly those of thirteen elements."* By Dr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of September 26, 1908).

In a paper to this Academy [1]) Prof. J. DE VRIES gave a triple system of 13 elements of a different type than the cyclic system of Prof. NETTO [2]); he added however the observation, that no proof has been furnished of these types being the only ones.

Mr. K. ZULAUF shows in his dissertation [3]) that the systems given formerly by KIRKMAN (1853) and REISZ (1859) are identical to that of DE VRIES, so that the number of *known* systems is *two*; neither is anything here decided about the number of *possible* systems.

It seemed desirable to decide upon this point by means of a special investigation [4]). To this end some facility is offered by using those expressions which are used in the theory of the configurations, by regarding the 13 elements as *points*, the 26 triplets as *lines* which bear three of the points; the whole of the triple system then becomes the scheme of a diagonalless Cf. $(13_6, 26_3)$ where it is irrelevant whether this Cf. can be geometrically realized or not. A classification of these Cff. is now our aim in view.

The rest figure of the second order of a line of such a Cf., i.e. what remains if we leave out that line with its three points and the 3×5 lines passing through these points, is of necessity a Cf. (10_3), the 10 points of which are in three ways perspective and that according to the three points left out.

But then reversely each imaginable Cf. $(13_6, 26_3)$ of the desired type is obtained by

1st. starting from all possible Cff. (10_3),

2nd. by constructing for each Cf. (10_3) the Cf. $(10_3, 15_2)$ of its diagonals,

1) *Versl. Kon. Akad. v. Wet.* III, p. 64, 1894.

2) *Substitutionentheorie*, p. 220; *Math. Annalen*, Vol. 42.

3) *"Ueber Tripelsysteme von 13 Elementen"*, Giessen, 1897.

4) I subsequently find this question treated also by DE PASQUALE (*Rendic. R. Ist. Lombardo*, 2nd Ser., 32, 1899).

3rd. by investigating for each of these Cff. $(10_3, 15_2)$ in how many ways its 15 lines break up into three *principal five-sides* [1]),

4th. by imagining for each of these cases the 5 lines of each principal five-side as converging to one point and these three points as collinear.

The number of types of Cff. (10_3) amounts, according to the classifications of KANTOR [2]) and SCHROETER [3]), to *ten*, to be distinguished

Those of the system of KIRKMAN are:

K																									
1	1	1	2	2	3	3	5	5	7	1	2	3	4	6	1	2	3	4	6	1	2	3	4	5	*a*
2	4	6	4	8	4	9	6	7	8	8	5	7	0	9	9	0	5	7	8	0	7	6	9	8	*b*
3	5	7	6	9	8	0	0	9	0	*a*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*c*	*c*	*c*	*c*	*c*	*c*
IX	VIII	V	V	V	V	VIII	V	IX	V	VIII	V	V	II	VIII	VI	VIII	V	VIII	IX	X	III	III	X	III	X

The Cff. of the diagonals of the 10 Cff. (10_3) are resp.

	II	III	IV	V	VI	VII	VIII	IX	X
1 8	1 8	1 8	1 8	1 8	1 8	1 8	1 8	1 8	1 8
1 9	1 9	1 9	1 9	1 9	1 9	1 9	1 9	1 9	1 9
1 0	1 0	1 0	1 0	1 0	1 0	1 0	1 0	1 0	1 0
2 5	2 5	2 5	2 6	2 6	2 5	2 5	2 5	2 5	2 5
2 7	2 7	2 6	2 7	2 7	2 7	2 7	2 9	2 6	2 7
2 0	2 0	2 0	2 0	2 0	2 0	2 0	2 0	2 4	2 9
3 4	3 4	3 4	3 4	3 4	3 4	3 4	3 5	3 7	3 5
3 6	3 6	3 7	3 5	3 5	3 6	3 6	3 6	3 8	3 6
3 0	3 0	3 0	3 0	3 9	3 9	3 8	3 7	3 0	3 7
4 7	4 7	4 7	4 7	4 7	4 7	4 7	4 7	4 7	4 7
4 9	4 9	4 9	4 9	4 0	4 9	4 9	4 8	4 9	4 8
5 6	5 6	5 6	5 6	5 6	5 6	5 6	4 9	5 6	4 9
5 9	5 8	5 8	5 8	5 8	5 8	5 0	5 7	5 0	5 0
6 8	6 8	6 9	6 9	6 8	6 8	6 8	6 8	6 8	6 8
7 8	7 9	7 8	7 8	7 9	7 0	7 9	6 0	7 9	6 0

1) By principal n-side of a Cf. MARTINETTI and DE VRIES understand a group of n Cf-lines which contain together once all the Cf-points.

2) *Wiener Sitzungsberichte* 84, 1881.

3) *Göttinger Nachrichten*, 1889, N°. 8.

1	1 1 1 8 2 3 2 3 4 5 1 2 3 5 7 1 2 4 3 6 1 2 3 4 5 *a* 2 4 6 9 4 7 6 5 6 7 8 0 4 6 9 9 5 7 0 8 0 7 6 9 8 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
2	1 1 1 8 2 3 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 6 7 *a* 2 4 6 9 4 7 6 5 6 7 8 7 0 9 6 9 0 6 7 8 0 5 4 8 9 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
3	1 1 1 8 2 3 2 3 4 5 1 2 3 4 6 1 2 3 5 7 1 2 3 4 5 *a* 2 4 6 9 4 6 7 5 6 7 8 5 0 7 9 9 0 4 6 8 0 6 7 9 8 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
4	1 1 1 8 2 3 2 3 4 5 1 2 3 4 5 1 2 3 4 5 1 2 3 4 7 *a* 2 4 6 9 4 6 7 5 6 7 8 0 7 9 6 9 6 0 7 8 0 5 4 9 8 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
5	1 1 1 8 2 3 2 4 3 5 1 2 3 5 7 1 2 3 4 6 1 2 3 4 5 *a* 2 4 6 9 4 7 5 6 6 7 8 0 4 6 9 9 7 5 0 8 0 6 9 7 8 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
6	1 1 1 8 2 3 2 4 3 5 1 [illegible] 3 4 5 1 2 3 4 6 1 2 3 5 7 *a* 2 4 6 9 4 7 5 6 6 7 8 7 9 0 6 9 0 5 7 8 0 6 4 8 9 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
7	1 1 1 8 2 3 2 4 3 5 1 2 3 4 5 1 2 3 4 6 1 2 3 5 7 *a* 2 4 6 9 4 7 5 6 6 7 8 0 9 7 6 9 7 5 0 8 0 6 4 8 9 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
8	1 1 1 8 2 3 2 5 3 4 1 2 3 4 5 1 2 3 6 7 1 2 3 4 5 *a* 2 4 6 9 4 7 6 7 5 6 8 0 9 7 6 9 5 4 8 0 0 7 6 9 8 *b* 3 5 7 0 8 8 9 9 0 0 *a a a a a b b b b b c c c c c c*
9	1 1 1 2 4 6 5 3 7 2 1 2 3 4 5 1 2 3 4 5 1 2 3 6 7 *a* 2 4 6 8 8 9 7 5 3 4 8 7 6 9 0 9 0 8 7 6 0 5 4 8 9 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
10	1 1 1 3 5 7 2 6 4 2 1 2 3 4 6 1 2 3 4 6 1 2 3 4 5 *a* 2 4 6 8 8 9 7 5 3 4 8 5 7 9 0 9 0 5 7 8 0 9 6 8 7 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
11	1 1 1 3 5 7 2 6 4 2 1 2 3 4 6 1 2 3 4 5 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 5 7 9 0 9 0 6 8 7 0 9 5 7 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
12	1 1 1 3 5 7 2 6 4 2 1 2 3 4 5 1 2 3 4 6 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 0 6 9 7 9 5 7 8 0 0 9 5 7 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
13	1 1 1 3 5 7 2 6 4 2 1 2 3 4 6 1 2 3 4 5 2 3 4 1 6 *a* 2 4 6 8 8 9 7 5 3 4 8 9 5 7 0 9 0 6 8 7 5 7 9 0 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
14	1 1 1 2 [illegible] 6 5 3 2 3 1 2 3 5 7 1 2 3 4 5 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 7 4 8 4 0 6 9 9 6 8 7 0 0 5 7 9 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
15	1 1 1 2 4 6 5 3 2 3 1 2 3 4 5 1 2 3 4 6 1 2 3 5 7 *a* 2 4 6 8 8 9 7 5 7 4 8 6 7 9 0 9 5 0 7 8 0 4 8 6 9 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
16	1 1 1 3 2 7 5 6 4 2 1 3 4 6 2 1 2 3 4 5 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 7 9 0 5 9 7 6 8 0 0 9 5 7 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
17	1 1 1 3 2 7 5 6 4 2 1 2 3 4 5 1 2 3 4 6 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 7 6 9 0 9 5 7 8 0 0 9 5 7 8 *b* 3 5 7 9 0 0 8 [illegible] 0 6 *a a a a a b b b b b c c c c c c*
18	1 1 1 3 2 7 5 6 4 2 1 2 3 4 6 1 2 3 4 5 1 2 5 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 9 5 7 0 9 7 6 8 0 0 5 7 9 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
19	1 1 1 3 2 7 5 6 4 2 1 2 3 4 5 1 2 3 4 6 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 9 6 7 0 9 5 7 8 0 0 7 5 9 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*
20	1 1 1 3 2 7 5 6 4 2 1 2 3 4 5 1 2 3 4 6 1 2 3 4 6 *a* 2 4 6 8 8 9 7 5 3 4 8 9 6 7 0 9 7 5 8 0 0 5 7 9 8 *b* 3 5 7 9 0 0 8 9 0 6 *a a a a a b b b b b c c c c c c*

according to their rest figures. If we use the list of SCHROETER, then the rest figures of the triplets of the system of NETTO are resp.:

N $(01..a..)_{cyc.}$ all VII, $(0..6.8..)_{cyc}$ [1]) all VIII.

Their possible decompositions into principal five-sides are:

I	none		
II	two,	see **1** and **2**	in the preceding table
III	two,	,, **3** ,, **4**	,, ,, ,,
IV	none		
V	three,	,, **5, 6, 7**	,, ,, ,,
VI	one,	,, **8**	,, ,, ,,
VII	one,	,, **9**	,, ,, ,,
VIII	four,	,, **10, 11, 12, 13**	,, ,, ,,
IX	two,	,, **14** and **15**	,, ,, ,,
X	five,	,, **16, 17, 8, 19, 20**	,, ,, ,,

According to 4 these decompositions now give rise to 20 triple systems, among which occur with certainty all possible systems; however, they can be identical mutually and to **K** or to **N**. We now remark that VI and VII can be completed in but one way; so as soon as these rests occur the system is identical to **8**, resp. **9**; from this follows already that **K** and **8**, and likewise **N** and **9** are of the same type (see table page 292).

Now rest figure VII appears for **11** according to the triplet 37*a*; VI on the contrary for:

1, 2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20

resp. according to the triplets:

123, 123, 145, 167, 123, 123, 47*a*, 37*a*, 35*c*, 57*b*, 56*a*, 79*c*, 49*a*, 27*a*, 18*a*, 35*c* and 60*b*;

so **11** is identical to **9** and **K**, the remaining are identical to **8** and **N** and we have proved:

that except the two known forms no triple systems of thirteen elements exist.

At the same time is evident that to recognise a given system it is sufficient to find a few rest figures until one meets either VII (or VIII in greater number than six) or II, III, V, VI, IX or X.

The preceding has moreover given a method [2]) to determine the

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[1]) The cyclic order is here: 0 1 2 3 4 5 6 7 8 9 *a b c*.

[2]) Compare DE VRIES *Versl. en Meded. Kon. Akad. v. Wet.* VI, p. 13, where in a more restricted sense the same method is used to trace the (exclusively *regular*) principal poly-sides of the Cff. $-_n$. Such a principal poly-side however determines a triple system and reversely. Indeed, the cyclic systems of 13 and 15 appear already in the work of DE VRIES (l. c. p. 16 and 17), however without being regarded as such.

number of triple systems for arbitrary n (if only $\equiv 1$ or 3, mod. 6), i.e. to classify all diagonalless

$$Cff. \left\{ n_{\frac{n-1}{2}} \ , \ \left(\frac{n(n-1)}{2\,.\,3}\right)_3 \right\},$$

namely :

1. by starting from all possible :

$$Cff. \left\{ (n-3)_{\frac{n-7}{2}} \ , \ \left(\frac{(n-3)\,(n-7)}{6}\right)_3 \right\},$$

2. by enumerating the Cff. of its diagonals, those are :

$$Cff. \left\{ (n-3)_3 \ , \ \left(\frac{3(n-3)}{2}\right)_2 \right\},$$

3. by investigating in how many ways each of these Cff. can break up according to its lines into three principal $\frac{n-3}{2}$ sides,

4. by every time assuming the lines of such a poly-side to be convergent to one point and these three points to be collinear,

5. by arranging the obtained systems in types.

Already for $n = 15$, however, this method is checked by the absence of the classification ot Cff. $(12_4, 16_3)$ necessary for 1, of which only some six forms have been enumerated[1]). Let us restrict ourselves to the best known and most regular form, of HESSE :

A	1	2	3	4
5	a	b	c	d
6	b	c	d	a
7	c	d	a	b
8	d	a	b	c

in which each point of the quadruplet $abcd$ is collinear to the points of the two other quadruplets in the same row of column, then *three* types of triple systems of 15 elements appear :

I. Complement $x\,a\,b$, $x\,c\,d$, $x\,12$, $x\,34$, $x\,56$, $x\,78$,
$y\,a\,c$, $y\,b\,d$, $y\,13$, $y\,24$, $y\,57$, $y\,68$,
$z\,a\,d$, $z\,b\,c$, $z\,14$, $z\,23$, $z\,58$, $z\,67$,
$x\,y\,z$.

All rests are A, the system is identical to the cyclic one[2]):
(1 . 2 . . 5 . . .) ; (1 . 3 . . . 9 . . .) ; (1 . . . 6 . . . 11 . . .).

II. Complement $x\,a\,b$, $x\,c\,d$, $x\,12$, $x\,34$, $x\,56$, $x\,78$,
$y\,a\,c$, $y\,b\,d$, $y\,13$, $y\,24$, $y\,58$, $y\,67$,
$z\,a\,d$, $z\,b\,c$, $z\,14$, $z\,23$, $z\,57$, $z\,68$,
$x\,y\,z$.

1) Comp. the author's dissertation; "Bijdragen tot de theorie der configuraties", Amsterdam 1907, § 36.

2) L. HEFFTER, *Math. Annalen*, Vol. 49.

There are three rests A, namely of

$$x\,y\,z, \quad x\,56 \quad \text{and} \quad x\,78,$$

there are four rests B, i. e. of the type of the Cf. $(16_3, 12_4)$ of DE VRIES of the composition

B -	1	2	3	4
5	a	b	c	d
6	b	c	d	a
7	c	d	a	b
8	d	a	b	c

These rests appear for $x\,a\,b$, $x\,c\,d$, $x\,12$, $x\,34$.

The other rests are of other types.

III, Complement $x\,a\,b$, $x\,c\,d$, $x\,12$; $x\,34$, $x\,58$, $x\,67$,
$y\,a\,c$, $y\,b\,d$, $y\,13$, $y\,24$, $y\,56$, $y\,78$,
$z\,a\,d$, $z\,b\,c$, $z\,14$, $z\,23$, $z\,57$, $z\,68$,
$x\,y\,z$.

Only $x\,y\,z$ has rest A, there are no rests B, all other rests are of other types.

Whether by completing other Cff. $(16_3, 12_4)$ other systems than the three above-mentioned will appear, remains undecided; at any rate B will lead in at least *one* way to II as it appears among its rests.

Anatomy. — "*About the function of the ventral group of nuclei in the thalamus opticus of man.*" By Prof. C. WINKLER and Dr. D. M. VAN LONDEN.

The following remarkable case was offered to our observation in the neurological clinical department of the Binnengasthuis.

An unmarried woman, aged seventy-seven, not having suffered previously of any serious illness, and somewhat dull of hearing during the last years, got one single but severe fit of dizziness three weeks before her admittance.

On Jan. 8th 1908 she was found in her room unconscious, and transported thence to the Binnengasthuis, on arrival there she was slightly wandering in her mind. This state continued for three days, afterwards she made no complaints either of pain or of paraesthesia. Incontinence never occurred. There was found sclerosis of the arteries and dilatation of the heart. The urine contained $^1/_4$ $^0/_{00}$ of albumen and hyaline-cylindres. After three months death ensued caused by