$$\left\| \begin{matrix} c_\lambda & x_1 q_\lambda & b_\iota^2 - x_1 q'_\tau \\ c'_\lambda & b_\lambda^2 - x_1 p_x & x_1 p'_\lambda \end{matrix} \right\| = 0 ;$$

by multiplying the terms of the first column by x_1 and by applying in any arbitrary way the addition of the rows and the columns, there is always a matrix of six quadratic forms annulling itself for the conic $x_1 = b_\iota^2 = 0$.

The projective theorems relating to the circular sextic γ_6 can thus be translated into properties of the most general sextic of genus two. It is superfluous to write down how these theorems run; we shall quote but one as an example: in every sextic of genus two the planes of the sexisecant conics pass through a fixed point of the quadrisecant.

5. If the intersection of the two cubic surfaces considered in the preceding (13) and (14) is completed by a line and a conic having a point in common, we have not a special case of the preceding case, in the sense of HALPHEN; but this line and this conic form a special case of a twisted cubic and the sextic is then of order three. Really in this case the equations (13) and (14) can be such that the line $x_2 x_3$ annuls $a_\iota^2, a'_2{}^2$ and b_ι^2 without annulling c_ι of c'_ι and the matrix

$$\left\| \begin{matrix} x_1 & c_\iota & c'_\iota \\ b_\iota^2 & a_\iota^2 & a_\iota'^2 \end{matrix} \right\|$$

has then the elements of its second row disappearing for a same line.

By this proceeding we can study the sextic of genus three; we can refind the univalent correspondence between its points and its trisecants, correspondence found by Mr. F. SCHUR (*Math. Ann.* vol. 18); we can bring back the representation of the curve to a matrix of twelve linear forms which we have studied in our *Cinq Etudes* and in the *Bulletins de l'Académie royale de Belgique* (May 1907), etc.

Ghent, Oct. 26, 1908.

Mathematics. — *"On the combinatory problem of* STEINER." By Dr. J. A. BARRAU. Communicated by Prof. D. J. KORTEWEG.

(Communicated in the meeting of October 31, 1908).

In its most general form this problem runs as follows:

for which values of n and in how many really different[1]) *ways is it possible to write down a number of combinations p to p of n elements in such a way that all combinations q to q appear in it, each one time?*

[1]) i. e. which do not pass into each other by means of substitutions of the n elements.

A geometrical way of putting the question is this:
which combinatory configurations whose points S_0 are represented by the combinations q to q of n letters, whilst the combinations p to p represent its Sp_{p-q}, possess systems of Sp_{p-q} containing all points of the Cf each one time and how many types of such systems appear in each definite case?

The first question gave rise to investigations for $=3$, $q=2$, the *triple systems* (KIRKMAN, REISS, NETTO, MOORE, HEFFTER, BRUNEL [1]); the second question is discussed by J. DE VRIES [2]) and CARP [3]).

Here some results are communicated for $p>3$, $q>2$. We adhere for this to the first form of the question and we call a system as is demanded there an $S(p, q), n$.

If we isolate in an $S(p, q), n$ all sets of p having in common a certain arbitrarily chosen letter, and if we omit that letter from it, an $S(p-1, q-1), n-1$ is generated; repetition of the operation gives rise to an $S(p-2, q-2), n-2$ and so on; the possibility of an $S(p, q)$ presupposes thus that of a series of systems of lower rank [4]), which series can be broken off at

$$S(p-q+2, 2), \; n-q+2$$

Inversely all imaginable systems are acquired by completion of systems commencing with $q=2$.

So we can expect:

out of $S(3,2), n=7$	:	$S(4,3), n=8$	A
out of $S(3,2), n=9$	:	$S(4,3), n=10$	B
		$S(5,4), n=11$	C
		$S(6,5), n=12$	D

etc. We will show that the four systems mentioned exist *each of them in one type.*

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1) *Cambridge and Dublin Math. Journal* II, 1847; *Journal f. d. r. u. a. Mathem.* 56, 1859; *Mathem. Annalen* 42, 1893; 43, 1893; 49, 1897; 50, 1898; *Association française, Congrès de Bordeaux* 1895; *Journal de Liouville* (5) VII, 1901.

2) *Versl. en Meded. Kon. Akad. v. Wet.* 3rd series, VI, p. 13, 1889; *Mathem. Annalen* 34, 35, 1889, '90.

3) Dissertation, Utrecht 1902, p. 38.

4) For *each* system of the series the condition must be satisfied: $\binom{n}{q}$ divisible by $\binom{p}{q}$. Thus there will be no $S(6,4), n=15$, although $\binom{15}{4}$ is divisible by $\binom{6}{4}$, on account of the impossibility of an $S(5,3)$ $n=14$: $\binom{14}{3}$ is not divisible by $\binom{5}{3}$.

A. An $S(4, 3)$, $n = 8$ consists of $\binom{8}{3} : \binom{4}{3} = 14$ quadruplets, so it is a schematic Cf. $(8_7, 14_4)$. If we add to the triplets of an $S(3, 2)$, $n = 7$, that is of a Cf. (7_3) a new letter, and if out of the quadruplets thus formed we choose one, then in the completion with seven new quadruplets sought for the pairs of the selected triplets must appear still twice; so a new quadruplet remains, complementary to the one selected.

This holds for each quadruplet; the whole completion is thus complementary and only possible in one way [1]).

B. If in an $S(4, 3)$, $n = 10$, that is a Cf. $(10_{12}, 30_4)$, we choose an arbitrary quadruplet 1 2 3 4, each pair of these letters appears in three more quadruplets; so there are 18 more such quadruplets. Then each single letter appears two times more, completed with triplets out of 5, 6, 7, 8, 9, 0, which triplets form thus together a Cf. $(6_4, 8_3)$. Of the whole system only three quadruplets out of 5, 6, 7, 8, 9, 0 remain, which in pairs may have at most only two letters in common. Such systems of three exist however only in one type [2]), as:

5 6 7 8 ; 5 6 9 0 ; 7 8 9 0,

with which the system to be formed must commence. The eight triplets of letters missing here form of necessity the Cf. $(6_4, 8_3)$, which breaks up only in *one* way into four pairs completing each other. If we complete these pairs respectively with 1, 2, 3 and 4 (in which order is irrelevant) the following quadraplets are formed:

1 5 7 9 ;	2 5 7 0 ;	3 5 8 9 ;	4 5 8 0
1 6 8 0 ;	2 6 8 9 ;	3 6 7 0 ;	4 6 7 9.

The entire further completion is now determined and *must* run as follows:

1 4 5 6 ;	1 2 5 8 ;	2 3 7 9 ;
2 3 5 6 ;	1 2 6 7 ;	2 3 8 0 ;
1 3 7 8 ;	1 3 5 0 ;	2 4 5 9 ;
2 4 7 8 ;	1 3 6 9 ;	2 4 6 0 ;
1 2 9 0 ;	1 4 7 0 ;	3 4 5 7 ;
3 4 9 0 ;	1 4 8 9 ;	3 4 6 8 .

Now that with this the existence of only *one* type is assumed, we can give it a simpler form; we shall do this in two ways.

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1) The scheme $(8_7, 14_4)$ indicates in the measure-polytope B_8 the vertices, forming with point zero and its opposite vertex together a cross-polytope C_4 (*Nieuw Archief v. Wisk.* 2nd Series, VII, p. 255).

2) The complementary type of a division of six elements into three pairs.

We in the first place remember that the $S(3, 2)$, $n = 9$ deduced from $S(4, 3)$, $n = 10$ is *regular commutative* (MOORE), i. e. that its group possesses a regular commutative subgroup of the type:

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5

Indeed, an $S(3, 2)$, $n = 9$ appears if we submit the triplets

1 2 3 ; 1 4 7 ; 1 5 9 ; 1 6 8

to all the substitutions of this group.

If we now add to each of these twelve triplets a zero and if we submit the quadruplets

1 2 4 5 and 1 2 6 9

to the substitutions of the group, then the $12 + 18 = 30$ quadruplets of the $S(4, 3)$, $n = 10$ are formed.

Secondly we observe, that the system is cyclic and appears among others by submitting the quadruplets

1 2 3 7 ; 1 2 4 5 ; 1 3 5 8

to the cycle (1 2 3 4 5 6 7 8 9 0).

C. If we choose out of an $S(5, 4)$, $n = 11$, that is a Cf. $(11_{30}, 66_5)$, a quintuple 1 2 3 4 5, then all triplets of it must appear still three times, all pairs moreover still two times, the single letters afterwards three times, with which the $10 \times 3 + 10 \times 2 + 5 \times 3 = 65$ remaining quintuplets of the system are exhausted. The single letters are completed with quadruplets out of 6, 7, 8, 9, 0, a, which may have at most three letters in common and which form together a Cf. $(6_{10}, 15_4)$, consisting of five $(6_2, 3_4)$ as appeared in *B*. Of this but *one* type exists[1]), so that e. g. the $S(5, 4)$ commences with:

1 8 9 0 a	2 7 9 0 a	3 7 8 0 a	4 7 8 9 a	5 7 8 9 0
1 6 7 0 a	2 6 8 9 a	3 6 8 9 0	4 6 8 0 a	5 6 9 0 a
1 6 7 8 9	2 6 7 8 0	3 6 7 9 a	4 6 7 9 0	5 6 7 8 a

[1]) Deduced from the well-known system of five three-divisions of six elements of SERRET.

The further constitution is now determined and must run as follows:

1 2 6 9 0	1 4 6 9 *a*	2 3 6 0 *a*	2 5 6 7 9	3 5 6 7 0
1 2 7 8 *a*	1 4 7 8 0	2 3 7 8 9	2 5 8 0 *a*	3 5 8 9 *a*
1 3 6 8 *a*	1 5 6 8 0	2 4 6 7 *a*	3 4 6 7 8	4 5 6 8 9
1 3 7 9 0	1 5 7 9 *a*	2 4 8 9 0	3 4 9 0 *a*	4 5 7 0 *a*

1 2 3 6 7	1 2 5 6 *a*	1 3 5 6 9	2 3 4 6 9	2 4 5 6 0
1 2 3 8 0	1 2 5 7 0	1 3 5 7 8	2 3 4 7 0	2 4 5 7 8
1 2 3 9 *a*	1 2 5 8 9	1 3 5 0 *a*	2 3 4 8 *a*	2 4 5 9 *a*
1 2 4 6 8	1 3 4 6 0	1 4 5 6 7	2 3 5 6 8	3 4 5 6 *a*
1 2 4 7 9	1 3 4 7 *a*	1 4 5 8 *a*	2 3 5 7 *a*	3 4 5 7 9
1 2 4 0 *a*	1 3 4 8 9	1 4 5 9 0	2 3 5 9 0	3 4 5 8 0

A simple form of the system [1]) is obtained by submitting the quintuplets

$$1\ 2\ 3\ 4\ 0;\quad 1\ 2\ 3\ 5\ 8;\quad 1\ 2\ 3\ 6\ 7;$$
$$1\ 2\ 4\ 5\ 9;\quad 1\ 2\ 4\ 6\ 8;\quad 1\ 2\ 5\ 7\ 0$$

to the cycle:

$$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 0\ a).$$

D. By a reasoning analogous to that in section *A* is evident that $S(6,5)$, $n = 12$, that is a Cf. $(12_{66}, 132_6)$ *must* appear by complementary completion of $S(5,4)$, $n = 11$, so it appears only in *one* type.

In general out of each

$$S(p = q + 1, q),\ n = 2q + 3$$

is formed by complementary completion an

$$S(p = q + 2, q + 1), n = 2q + 4.$$

Passing to systems $S(p, q = 2), n$ for arbitrary p [2]), we observe that for their existence is necessary at least:

$$\frac{n(n-1)}{1.2} \text{ divisible by } \frac{p(p-1)}{1.2},$$

and

$$n-1 \text{ divisible by } p-1,$$

which conditions are fulfilled only by the two series of numbers

$$\begin{array}{ll} I & n = p(p-1) \cdot x + 1 \\ II & n = p(p-1) \cdot x + p \end{array} \Bigg\} \; x = 0, 1, 2, \ldots\ldots$$

[1]) A $S(5,4)$, $n = 11$ was given by Lea (*Educ. Times* IX), the method in which this was generated is unknown to me, as this publication was not attainable for me; however, the system must be of the same type. We remark moreover, that to prove this, it is not sufficient to assert that all remainders have a fixed type, comp. Martinetti (*Annali di Matem.* (2) XV). The same holds for *A*, *B* and *D*.

[2]) These $S(p,2)$ are to be distinguished from the systems of Brunel, in which each set of p is regarded as containing only the p pairs of elements succeeding each other (*Proc. Verb. Soc. de Bordeaux*, 1895/96, p. 58 and 1898/99, p. 59, 71).

The term $x=0$ has no significance in I; in II it indicates that when writing all $n=p$ letters we have written at the same time all pairs of those letters. This system $II, x=0$ will be of service in what follows.

We shall now give a more extensive form to the *multiplication-theorem* of NETTO (l.c. § 3) which becomes:

Out of an $S(p,2)\,n_1$ and an $S(p,2), n_2$ an $S(p,2)\,n_1 n_2$ can be deduced, if one disposes of a system of $p\,(p-1)$ permutations of p elements, in which a couple of elements never occupies the same place.

For, if the elements of the two systems are resp. $a_k\,(k=1,2\ldots n_1)$ and $b_l\,(l=1,2,\ldots\ldots n_2)$ we can then designate $n_1 n_2$ new elements by $c_{k,l}$. Of these elements we form *three* kinds of p sets of p, namely:

1[st]. out of *each* set of p $a_1\,a_2\ldots\ldots a_p$ new p-sets:

$$c_{1,l},\, c_{2,l},\, \ldots,\, c_{p,l}\ \ (l=1,2\ldots\ldots n_2);$$

2[nd]. out of *each* set of p $b_1 b_2\ldots\ldots b_p$ new p-sets:

$$c_{k,1},\, c_{k,2},\, \ldots\ldots\, c_{k,p}\ \ (k=1,2\ldots\ldots n_1);$$

3[rd]. out of *each* set of p $a_1 a_2\ldots\ldots a_p$, combined with *each* set of p $b_1 b_2\ldots\ldots b_p$ new p-sets:

$c_{1,l_1}, c_{2,l_2}\ldots\ldots c_{p,l_p}$, where $l_1\ldots\ldots l_p$ are every time the same as the indices $1\ldots\ldots p$ of the set of p of b, yet differ $p-1$ times in order of succession according to the $p-1$ permutations of the system of permutations supposed as disposable.

It is clear that in this way a couple of the new elements can never appear more than once whilst the number of formed sets of p amounts to:

$$n_2\frac{n_1(n_1-1)}{p(p-1)}+n_1\frac{n_2(n_2-1)}{p(p-1)}+p\,(p-1)\frac{n_1(n_1-1)}{p(p-1)}\cdot\frac{n_2(n_2-1)}{p(p-1)}=\frac{n_1 n_2(n_1 n_2-1)}{p(p-1)},$$

so that really an $S(p,2), n_1 n_2$ is formed.

Now we dispose of such a permutation-system, when:

1) $p=4$: the twelve even permutations;

2) p *prime*, the system then consists of:

$$\begin{array}{l}(1,\ 2,\ 3,\ 4\ldots\ldots\ p)_{cyc.}\\(1,\ 3,\ 5,\ 7\ldots\ldots\ p-1)_{cyc.}\\(1,\ 4,\ 7,\ 10\ldots\ldots)_{cyc.}\\\ldots\ldots\ldots\ldots\ldots\ldots\\(1,\ p,\ p-1,\ p-2\ldots 2)_{cyc.}\end{array}$$[1].

[1]) Comp. e.g. BRUNEL, *Proc. Verb. Soc. de Bordeaux* 1894/95, p. 56, or AHRENS. *Mathematische Unterhaltungen*, p. 272: "Promenaden von n^2 Personen zu je n". It is not decided whether also other values of p allow a solution.

(For $p=3$ the system just contains the total group and so the theorem of NETTO reappears).

So we have:

The possibility of the multiplication is assured for $p=4$ or prime.

Now by taking as factors $n_1=n_2=p$, so term $II, x=0$, we obtain:

For $p=4$ or prime the existence of $S(p,2), n=p^2$ is assured.

Such a system, term II, $x=1$ is, regarded as Cf.-scheme, a Cf. $\{p^2{}_{p+1}, p(p+1)_p\}$; its p-sets can be divided into $(p+1)$ principal groups of p, each of which contains all the p^2 elements.

We can now give in a more extensive from an other theorem of NETTO (l.c. § 2), which becomes here:

Out of an $S(p,2), n$ an $S(p,2), (p-1)n+1$ can be formed if we have at our disposal a scheme of $(p-1)^2$ sets of p out of $p(p-1)$ elements, having mutually not more than one element in common, which elements must be able to break up into p principal groups of $(p-1)$.

For, we can add to the elements $a_{1,1}, a_{1,2} \ldots . a_{1,n}$ of the given system $(p-2)$ series of new ones:

$$\begin{array}{l} a_{2,1} \quad , a_{2,2} \quad , \ldots . a_{2,n} \\ \ldots \ldots \ldots \ldots \ldots \ldots \\ a_{p-1,1} , a_{p-1,2} , \ldots . a_{p-1,n} , \end{array}$$

and moreover a last element a_0.

For each set of p of the given system we must now form out of the $p(p-1)$ elements with the same second indices a scheme as the one indicated in the theorem, taking care that always the $(p-1)$ elements with equal second index form a principal group. Finally we must add to each principal group the element a_0, by which also these principal groups are completed to p. It is clear that the sets of p formed in this way can have two by two at most but *one* element in common, while their number amounts to:

$$(p-1)^2 \cdot \frac{n(n-1)}{p(p-1)} + n = \frac{\{(p-1)n+1\}\{(p-1)n\}}{p(p-1)},$$

so that an $S(p,2), (p-1)n+1$ is formed.

The possibility of the method now depends on the presence of a scheme:

$$\{p(p-1)_{p-1} , (p-1)^2_p\}$$

or, replacing p by $p-1$, of

$$\{p(p+1)_p \quad , p^2_{p+1}\},$$

but that is just the notation of the above-mentioned $S(p,2), n=p^2$, if but in the diagram we exchange rows and columns. It satisfies, moreover the further conditions; so we have:

The deduction of an $S(p,2)$, $(p-1)n+1$ *out of an* $S(p,2), n$ *is assured, when* $p=5$ *or a prime number* $+1$.

If we apply this to $n=p$, we find:

The existence of $S(p,2)$, $n=p(p-1)+1$ *is assured for* $p=5$ *or a prime number* $+1$.

These systems form the term I, $x=1$; as Cf.-schemes they are Cff. $\{p(p-1)+1_p\}$.

It is clear that their remainders to *one* element again become $S(p-1,2)$, $n=(p-1)^2$, that is term II, $x=1$ out of the series $p-1$.

In cyclic form we find:

$n=13$, $p=4$: (1, 2, . . 5, . . 7) cyc.
$n=21$, $p=5$: (1, 2, . . 5, . . 15, . . 17 . .) cyc.
$n=31$, $p=6$: (1, 2, . . 5, . . 7 . . 14, . . 22 . .) cyc. [1]

We finally pass on to the generation of an $S(4,2,)$ $n=25$.

As Cf. the system is a Cf. $(25_8, 50_4)$, the remainder to each quadruplet is a Cf. (21_4). The latter must have the property that the non-united elements may be united to triplets, so that out of it arises a Cf. $(21_4, 28_3)$ which breaks up into four principal-7-sides [2]. By imagining the seven lines of such a 7-side to be every time convergent to *one* point and the four points of convergence to be collinear, the desired Cf. $(25_8, 50_4)$ is formed.

We obtain a solution by submitting

1, . . 3, . . 9, . . 12;
1, . . 8, . . 15 , (completed by 22);
1, . . 2, . . 6 , (in turns completed by 23, 24 and 25);

to the cycle

(1, 2, 3, . . . 21)

and by finally adding:

22, 23, 24, 25.

In like manner we shall find that in general an

$$S(p,2),\ n=2p(p-1)+1,$$

that is a Cf. $\{2p(p-1)+1_{2p},\ 4p(p-1)+2_p\}$,

1) Comp. E. Malo, *(Interm. des Mathém.* XVI, p. 63). The $2p$ cycles given there for each p are mutually identical and so they furnish every time but one type.

2) Comp. a former paper in these Proceedings (p. 290). In the mean time I have found that the result given there, in as far as it concerns $n=13$ is obtained in about the same way by Brunel (*Journal de Liouville*, 1901), and already in 1899 by de Pasquale, *Rendic. R. Inst. Lombardo* (2) 32, p 213.

will consist of a

$$\text{Cf. } \{(2p-1)(p-1)_p\},$$

whose missing pairs of elements can be united to sets of $(p-1)$, so that a

$$\text{Cf. } \{2p(p-1)_p,\ p(2p-1)_{p-1}\}$$

is formed which breaks up into p principal-$(2p-1)$-sides.

As in the series $p=4$, just as for $p=3$, the two extensions of the theorems of NETTO can be applied, we can form, in the first set of hundred, systems $S(4, 2)$ for:

$$n = 13,\ 25,\ 49;\ 4,\ 16,\ 40,\ 52,\ 64,\ 76,\ 100.$$

The still missing values are:

$$n = 37,\ 61,\ 73,\ 85,\ 97;\ 28,\ 88,$$

of which 85 might be acquired by 28.

Physics. — *"Remarks on the Leyden observations of the Zeeman-Effect at low temperatures."* By W. VOIGT. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the Meeting of October 31, 1908).

The observations of KAMERLINGH ONNES and JEAN BEQUEREL [1]) on the ZEEMAN-Effect at exceedingly low temperatures have led to some surprising results, among which are two, very interesting from the theoretical standpoint. These I will try to throw light upon in the following.

I. It has been found, when the observation was made along the optic axis of an uniaxial crystal (where such a body acts as if isotropic) a longitudinal magnetic field being used, that the components of the ZEEMAN-doublet have different intensities. The component on the side of the shorter wave-length had generally (though not always) the greater intensity.

This result seems to show, that in the crystal one sense of rotation is preferred to the other. Therefore it might be interesting, to consider firstly the effect of a magnetic field on a naturally active crystal. I may at this point remark that, contrary to what should be expected, the effects of the natural activity and of the magnetic field do not superpose each other, but rather singular combined effects appear in the neighbourhood of an absorption band.

1) J. BECQUEREL and KAMERLINGH ONNES. These Proceedings February 29th 1908.