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Mathematics. — "*On fourdimensional nets and their sections by spaces.*" (Third part.) By Prof. P. H. SCHOUTE.

The net (C_{16}).

1. In the first part of this investigation we have found that the net (C_{16}) of cells $C_{16}^{(2\sqrt{2})}$ is formed out of three equally strongly developed groups of homothetic cells $C_{16}^{(2\sqrt{2})}$, *one* group of *erect* cells $C_{16}^{(2\sqrt{2})}$ polarly inscribed in eightcells $C_8^{(4)}$ and *two* groups of *inclined* cells $C_{16}^{'(2\sqrt{2})}$ and $C_{16}^{''(2\sqrt{2})}$ bodily inscribed in eightcells $C_8^{(2)}$, namely the *positive* group and the *negative* one. If we restrict ourselves once more, with respect to this net, to the sections by spaces normal to one of the four different axes of one of the sixteencells, it is evident from the table of connections between these different axes given in the first part (p. 544) — if we bear in mind that the three groups of cells of the net are equivalent — that we have only to consider three series of parallel intersecting spaces, viz. those normal to one of the axes OR_8, OF_8, OK_8 of the circumscribed eightcells. In these three cases, corresponding successively to the fifth, the fourth, and the third line of the quoted table, we find indeed for the erect sixteencells series of spaces normal to OE_{16}, OK_{16} and OF_{16}, whilst the first of the three cases provides us, for the inclined sixteencells, with a series of spaces normal to OR_{16}. So all in all we have to bring to light three different threedimensional space-fillings and their transformation connected with a parallel motion of the intersecting space. But in order to do this we have to consider more than the four usual series of intersecting spaces respectively normal to an axis OE_{16}, OK_{16}, OF_{16}, OR_{16}; for the spaces normal to OK_8 presenting themselves in the last of the three cases are not normal to any one of the four axes of the *inclined* sixteencells but to the line connecting the centre of one of these cells with the point characterized by the coordinates (3, 1, 1, 1) with respect to the system of coordinates with the four axes OE_{16} of that cell as axes. So all in all we have to deal with five series of parallel intersecting spaces which may be characterized by the symbols (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1), (3,1,1,1), as they are always normal to the diameter of the cell passing through the point the coordinates of which with respect to the axes OE_{16} of the cell are given by the corresponding symbol.

2. In the same way as we have done this in our second com-

munication for the six series of parallel sections of the eightcell we indicate the results of the intersection of a single sixteencell $C_{16}^{(2\sqrt{2})}$ in two different manners. A first plate will give the projections of the limiting elements of the sixteencell on the diameter normal to the intersecting spaces, which will enable us to deduce the sections from it tabularly; a second plate will give the sections themselves in parallel perspective, included in the sections with the polarly circumscribed $C_8^{(4)}$ or the bodily circumscribed $C_8^{(2)}$. Moreover a third plate will contain two groups of diagrams, the first of which will elucidate the manner of deduction of the projections given on plate I, whilst the second is concerned with the space-fillings obtained by the intersection of the net (C_{16}). In order to facilitate the survey of these space-fillings we deviate from the way followed in the second communication and treat together the more or less regular space-fillings presenting themselves here, instead of joining each of them separately to the corresponding generating three series of intersecting spaces.

We now first consider the four diagrams of the first group of plate III dominating the deduction of the projections of plate I. In fig. 1 we once more show how the inclined cell $C_{16}^{(2\sqrt{2})}$, indicated by its vertices only, is inscribed in the cell $C_8^{(2)}$. If we indicate by A, B, C, D the vertices of one of the sixteen limiting bodies, by A', B', C', D' the opposite ones, the sixteen limiting tetrahedra are

		$A'B'CD$		
	$A'BCD$	$A'BC'D$	$AB'C'D'$	
$ABCD$	$AB'CD$	$A'BCD'$	$A'BC'D'$	$A'B'C'D'$
	$ABC'D$	$AB'C'D$	$A'B'CD'$	
	$ABCD'$	$AB'CD'$	$A'B'C'D$	
		$ABC'D'$		

Of these five groups of 1, 4, 6, 4, 1 tetrahedra those of the first, the third, and the fifth groups are inscribed in the eight limiting cubes of the eightcell, whilst the four vertices of each of the tetrahedra of the second and the fourth group always split up with reference to two opposite limiting cubes of the eightcell into one vertex and three vertices.

With reference to the system of coordinates $O(Y_1, Y_2, Y_3, Y_4)$ of the four diagonals AA', BB', CC', DD' already used in the first communication the five series of parallel sections are characterized by the symbols (1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1), (3,1,1,1) mentioned above, from which can be deduced that the octuple of vertices of the sixteencell projects itself on the chosen axis of projection in these five cases in arrangements indicated by (1, 6, 1), (2, 4, 2), (3, 2, 3), (4, 4), (1, 3, 3, 1). Of these five cases the first and the fourth are evident by themselves; so the parts of plate I bearing the headings (1, 0, 0, 0) OE_{16} and (1, 1, 1, 1) OR_{16} can be understood immediately. The three other cases can be explained by the three following diagrams, a common characteristic of which is that the eight vertices of the sixteencell have been obtained by starting from the cube that is found by intersecting the eightcell of fig. 1 by the central space normal to AA', by splitting up the eight vertices of that cube into the two sets of vertices of bodily inscribed tetrahedra and by erecting normals on the space bearing that cube — i. e. by drawing in the diagram in parallel perspective lines parallel to AA'' — the length of which is equal to $\frac{1}{2} AA''$, in the vertices of one of the tetrahedra to one side and in the vertices of the other tetrahedron to the opposite side. This representation of the cube with the two quadruples of points $ABCD$, $A'B'C'D'$ has been repeated in three different positions by a motion parallel to itself from left to right and from above to below over the same distance, which gives rise to the three diagrams 2, 3, 4 which we will now examine one after another.

In fig. 2 the eight vertices of the sixteencell have been projected on to the line $F_8F'_8$, joining the midpoints of two opposite edges of the cube and forming therefore an axis OF_8 of the eightcell; on this axis the vertices A, B project themselves in F_8, the vertices C, D, C', D' in O and the vertices A', B' in F'_8. We find again here that the axis OF_8 of the eightcell is at the same time an axis OK_{16} of the bodily inscribed sixteencell, as F_8 is the midpoint of AB, and deduce now from the arrangement (2, 4, 2) of the projections of the vertices all that is indicated on plate I under the heading (1, 1, 0, 0) OK_{16}.

In fig. 3 the centres of gravity F_{16}, F'_{16} of the opposite faces ABC, $A'B'C'$ have been determined, and the axis OF_{16} joining these points forms the axis of projection. Then the three vertices A, B, C project themselves in F_{16}, the two vertices D, D' in O, the three vertices A', B', C' in F_{16}. From the arrangement (3, 2, 3) can then be deduced what appears in plate I under the heading (1, 1, 1, 0) OF_{16}.

In fig. 4 the diagonal $D''D''$ of the cube, forming an axis OK_8 of the eightcell, appears as axis of projection. Here the projection of the eight vertices of the sixteencell on that line $D''D'''$ is found in the easiest way by projecting these points first on to the space of the cube with the diagonal $D''D'''$ and by repeating this for the eight projections obtained with respect to the line $D''D'''$. For, the projections of the eight vertices of the sixteencell on to the space of the cube are the vertices of that cube, and these project themselves on $D''D'''$, if this line is divided by P and P' into three equal parts, according to the arrangement $(1, 3, 3, 1)$ in the points D'', P, P', D'''. From this arrangement $(1, 3, 3, 1)$ of the vertices can be deduced immediately what appears on plate I under the heading $(3, 1, 1, 1)OK_8$.

For each of the five cases considered we repeat under the heading "type" the manner in which the four couples of opposite vertices of the eightcell project themselves on the different axes.

3. We now proceed to the description of the sections, represented in parallel perspective on plate II, of the erect $C_{16}^{(2\sqrt{2})}$ and its envelope $C_8^{(4)}$ on one side, and the inclined $C'^{(2\sqrt{2})}_{16}$ and $C''^{(2\sqrt{2})}_{16}$ and their envelopes $C_8^{(2)}$ on the other. The sections of the C_{16} can be deduced from the tables of projection of plate I, those of the circumscribed $C_8^{(4)}$ and $C_8^{(2)}$ have already been given on plate II of the second communication.

By two thick vertical lines *this* plate II is divided into three parts, respectively related to sections normal to OR_8, normal to OF_8, normal to OK_8. Each of these three parts is divided by a thin vertical line into two columns; of these two columns the lefthand one always contains three sections of erect sixteencells, the right-hand one five or more sections of inclined sixteencells. We now consider separately each of the six columns so formed.

Sections normal to OR_8.

a. *Erect cells.* This case is the simplest of all. If by a motion parallel to itself of the intersecting space normal to the axis OE_{16} of $C_{16}^{(2\sqrt{2})}$ the point of intersection of that space with that axis moves from one of the two vertices situated on that axis to the other, the section with the circumscribed $C_8^{(4)}$ remains a cube with edge four and the section with the inscribed $C_{16}^{(2\sqrt{2})}$ itself, which is always a regular octahedron, increases in size from a point, the centre of the cube, to the inscribed octahedron with edge $2\sqrt{2}$ and then it passes through the same stadia in inverse order. Of the three diagrams the second represents an intermediate stadium, in which the edge of the octahedron is $\sqrt{2}$.

b. *Inclined cells.* If the point of intersection of the intersecting space normal to the axis OR_{16} of $C_{16}^{(2\sqrt{2})}$ with this axis describes this axis completely, the section with the bodily circumscribed $C_8^{(2)}$ remains a cube with edge two, whilst the section with the inscribed $C_{16}^{(2\sqrt{2})}$, always a tetrahedron truncated at the vertices and at the edges, transforms itself from a right tetrahedron to a left one in the manner shown by the five diagrams. In the third of the five we recognize the semiregular body (with regular faces) forming the combination of cube and octahedron in equilibrium, whilst the form represented by the second and the fourth show how this combination is formed out of the right tetrahedron and passes into the left one [1]).

Sections normal to OF_8.

a. *Erect cells.* Here a difference arises with respect to the fraction indicating the position of the intersecting space, according as the line through O normal to the intersecting space is considered either as an axis OK_{16} of the inscribed $C_{16}^{(2\sqrt{2})}$ or as an axis OF_8 of the circumscribed $C_8^{(4)}$. Therefore to each of the three diagrams presenting themselves here correspond two fractions, one below at the righthand side referring to the axis OK_{16}, another above at the lefthand side referring to the axis OF_8. If the point of intersection of the intersecting space with the axis OF_8 of $C_8^{(4)}$ describes this axis completely, the height of the rectangular parallelopipedon forming the section with $C_8^{(4)}$, the base of which is a square with side four, increases from nought to $4\sqrt{2}$ and then again decreases to nought. But only at the moment that this height is increased to $2\sqrt{2}$ does the polarly inscribed $C_{16}^{(2\sqrt{2})}$ begin to be cut. So we find in the three cases, where

[1]) For the sake of clearness the limiting elements of the section of the cell C_{16}' situated in the faces of the section of the enveloping box C_8 have been brought to the fore by indicating the vertices situated in *all* the faces of that envelope as black points, and by shading the faces of the section of the cell C_{16} situated in *visible* faces of that envelope.

the height is respectively $2\sqrt{2}$, $3\sqrt{2}$, $4\sqrt{2}$ and the fractions above to the left are $\frac{2}{8}, \frac{3}{8}, \frac{4}{8}$, for the fractions below to the right $0, \frac{1}{4}, \frac{2}{4}$ and for the sections of $C_{16}^{(2\sqrt{2})}$ an edge, a cube covered at two opposite faces by square pyramids, a square double-pyramid.

b. *Inclined cells.* In the five cases corresponding to the fractions $0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$ the section of the circumscribed $C_8^{(2)}$ is a rectangular parallelopipedon, the base of which is a square with side two, with a height $0, \frac{1}{2}\sqrt{2}, \sqrt{2}, \frac{3}{2}\sqrt{2}, 2\sqrt{2}$ successively. The sections of the inscribed $C_{16}^{(2\sqrt{2})}$ represented in the first, the third and the fifth of the five figures are equal to those of the preceding column and in the second and the fourth intermediate forms between these; in general the section can be characterized as a rectangular parallelopipedon with a square as base and upperplane, covered at these two faces by square pyramids, the faces of which have a determined inclination.

Sections normal to OK_8.

a. *Erect cells.* Here too, a difference presents itself as to the fractions, according as the diameter normal to the intersecting space is considered either as an axis OF_{16} or as an axis OK_8. If the point of intersection of the intersecting space with the axis OK_8 of $C_8^{(4)}$ describes that axis completely, the base of the prismatic section, the height of which remains four, transforms itself in the same manner as the section of a cube with edge four by a plane normal to a diagonal, and now at the moment that this base is increased to a triangle with side $4\sqrt{2}$ a face of the inscribed $C_{16}^{(2\sqrt{2})}$ appears in the intersecting space. So, to the fractions $\frac{4}{12}, \frac{5}{12}, \frac{6}{12}$ above to the left, correspond the fractions $0, \frac{1}{4}, \frac{2}{4}$ below to the right; so we find in the first diagram a triangle in a triangular prism, in the third a regular hexagonal double-pyramid in a regular hexagonal prism, in the second a form (12, 24, 14) bounded by two equilateral triangles, six isosceles triangles, six isosceles trapezia in a semiregular hexagonal prism regular as to the angles.

b. *Inclined cells.* The seven cases corresponding to the fractions $0, \frac{1}{12}, \frac{2}{12}, \ldots, \frac{6}{12}$ are all represented here. In the case corresponding to nought the section with $C_8^{(2)}$ is a line, here a vertical one, the section with $C_{16}^{(2\sqrt{2})}$ a point, here the upper extremity of that line. In the cases $\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}$ we find an irregular octahedron, inscribed in a triangular prism, bounded by two equilateral triangles of different size and two sets of three isosceles triangles of different form; the smaller of the two equilateral triangles is always inscribed in the upperplane of the prism, whilst the larger forms a normal section of the prism, successively at the height $\frac{3}{4}, \frac{2}{4}, \frac{1}{4}$, 0. Finally in the cases $\frac{5}{12}, \frac{6}{12}$ we find a semiregular hexagonal prism regular as to the angles and a regular one in which polyhedra (12, 24, 14) are inscribed, once more bounded by two equilateral triangles, six isosceles triangles and six isosceles trapezia. But here, in opposition to the form (12, 24, 14) found above, the two equilateral triangles instead of being homothetic have an opposite orientation. [1])

4. Before we pass to the generation of more or less regular space-fillings by intersecting the net (C_{16}) we wish to say a single word about the diagonal planes appearing in the sections of the cell C_{16} represented on plate II. In my communication "On groups of polyhedra with diagonal planes, derived from polytopes" published in these *Proceedings* of October (p. 277—290) it has been explained that any space intersecting C_{16} and not passing through one of the edges intersects this cell in a polyhedron with the property that through any edge of it passes one and only one diagonal plane, and that we only can obtain sections, through one or more edges of which pass two diagonal planes, if we choose an intersecting space passing through one or more edges of C_{16}. We have especially to show here

[1]) In the paper "Regelmässige Schnitte und Projectionen des Achtzelles u.s.w." (Regular sections and projections of the eightcell, the sixteencell, etc.", Verhandelingen of Amsterdam, first section, vol. II, N°. 2, 1894), I restricted myself principally to central sections; I only added incidentally a remark about the sections by spaces not passing through the centre. The figures 11 and 13 of that former paper, being not quite correct, should be replaced by the second figures of the third and fifth columns of plate II of this study.

why this particularity — as was already stated there — does not present itself in any of the sections of the four principal groups.

A mere inspection of plate II is sufficient to show, that *all* the sections of C_{16} represented there — not only those related to the axes OE_{16}, OK_{16}, OF_{16}, OR_{16} but also those of the last column — agree with one another in this, that any edge is situated in one and only one diagonal plane, moving parallel to itself if the intersecting space displaces itself parallelly. As an example we fix our attention on the figures of the fourth column, where a hexagon $MABNCD$ starts on half its journey as a line MN to end it as a lozenge $MANC$.

Now the reason, *why* no edge situated in two diagonal planes occurs here in the cases of sections by spaces containing edges of C_{16}, can be derived from plate I. It comes to this, that spaces through edges of C_{16}, not leaving that cell entirely on one side, do not present themselves for sections normal to OR_{16} or OF_{16}, that they pass through the centre O for sections normal to OE_{16} or OK_{16} and contain a face of C_{16} for sections under the heading $(3,1,1,1)OK_8$. If the intersecting space — see fig. 2 of the communication of October — contains not only the edge AB but also the centre O of the cell, the two points of intersection S_{13} S_{24} coincide in O, and instead of two diagonal planes ABS_{13}, ABS_{24} we find only one diagonal plane ABO, containing also the edge $A'B'$ opposite to AB and therefore intersecting the section in a square; this happens in the cases of the last figures of the first and the third column of plate II, for the first column with each, for the third column with only one diagonal plane, represented horizontally. In the case of the last column corresponding to the fractional symbol $\frac{4}{12}$ the triangle OPQ forming the base of the section is a face of C_{16}; so through any side of this triangle passes only one diagonal plane.

5. In order to determine the threedimensional space-fillings generated by intersection of the net (C_{16}) we can follow different ways, some of which are of a more theoretic, others of a more practical character. Those of the first group correspond in this, that we deduce from the section of a determined C_{16} with the intersecting space how this space must affect the other cells of the net (C_{16}). So we can project the axes of all the cells, normal to the intersecting space, on the axis taken as axis of projection, and deduce from the fraction corresponding to the chosen C_{16} the fraction corresponding to any other cell of the net; this method has been applied to the

net (C_8) in the second communication, and it can be of great service here, as the cells $C_8^{(2)}$ bodily circumscribed to the inclined cells C_{16} form a net (C_8). However, it often proves to be more practical to start from any section presenting itself, to hunt for other sections, possible in the position of the intersecting space under consideration, admitting a face agreeing in shape and in size with one of the faces of the chosen section, and to investigate if it is possible to arrive in this manner at a space-filling either by these two polyhedra only or by means of still more forms equally possible.

Space-fillings normal to OR_3. Let us imagine in threedimensional space a net of cubes with edge two, built up by cubes alternately white and black so as to form a threedimensional chessboard, with an infinite number of cubes, and let us describe in all white cubes a righthanded, in all black cubes a lefthanded tetrahedron. Then the interstitial spaces between these tetrahedra can be filled up by regular octahedra, forming with the tetrahedra the mixed net of tetrahedra and octahedra with common length of edge $2\sqrt{2}$. If we describe in all white cubes the tetrahedra truncated at vertices and edges of the second, in all black cubes the tetrahedra truncated at vertices and edges of the fourth of the five figures of the second column of plate II, the interstitial spaces can be filled up by regular octahedra of two different sizes, i.e. with edges $\frac{1}{2}\sqrt{2}$ and $\frac{3}{2}\sqrt{2}$. If we describe in *all* cubes the combination of cube and octahedron in equilibrium represented by the third of the five figures, the remaining interstitial spaces can be once more filled by regular octahedra of the same size, this time with the edge $\sqrt{2}$. These generally known results are obtained immediately by means of the method of juxtaposition, if we only bear in mind that two bodily inscribed sixteencells, the boxes $C_8^{(2)}$ of which have a limiting cube in common, are cut by any space normal to the space of that cube in polyhedra being one anothers mirror-image with respect to the plane of intersection as mirror, from which it ensues immediately that of the five figures of the second column the first and the fifth correspond to one another, also the second and the fourth, whilst the third stands for itself. By the juxtaposition, which comes here to the filling up of the interstitial spaces, we then find that the two extreme figures of the second column are to be combined with the two extreme figures of the first column, that the middle figure of the second column demands the middle figure of the first column, whilst the two remaining figures of the second column correspond to two intermediate sections with the fractions $\frac{1}{8}$ and $\frac{3}{8}$ of the first column, not represented here.

If the point O (fig. 5) is the centre and OR an axis OR_8 of one of the cells $C_8^{(2)}$ and we assume on the line which is to be considered as axis of projection a scale division with O as origin and half the edge of $C_8^{(2)}$ as unit, the vertices of the cells $C_8^{(2)}$ — and therefore also the centres of $C_8^{(4)}$ — project themselves into the points with a distance from O equal to an odd number of integers. If now the projection P of the intersecting space on to this axis lies between the origin and the point 1, and if $1-2x$ represents the distance OP, the section of all the cells $C_8^{(2)}$ corresponds to the fraction x, whilst both the series of $C_8^{(4)}$, the centres of which project themselves into the points -1 and $+1$, correspond to the fractions $y = \frac{x}{2}$ and $y' = \frac{x+1}{2}$. Now as the fraction x of a positively inscribed C_{16} inverts its sign and passes therefore into $1-x$, if this C_{16} is replaced by a negatively inscribed one, the fractions x and $1-x$ of the five figures of the second column belong together and to them correspond the fractions $\frac{1}{2}x$ and $\frac{1}{2}(x+1)$ of the first column. This result is in accordance with the preceding one; moreover it proves that it is preferable to say that the intermediate sections, not represented in the first column, corresponding to the second and the fourth figures of the second column, bear the fractional symbols $\frac{1}{8}$ and $\frac{5}{8}$.

It goes without saying that by the last method is indicated at the same time what the space-filling corresponding to an arbitrary value of x looks like; as this is immediately clear by itself we do not enter into details.

Space-fillings normal to OF_8. The result found above — that of the five figures of the second column those at the same distance from the middle one belong together — holds for this case too. This is proved easily, in a manner independent of preceding considerations, as follows. If PX_1, PX_2, PX_3, PX_4 (fig. 6) are the four edges of a $C_8^{(2)}$ meeting in a vertex P, if PQ and PR are the squares described on X_1PX_2 and X_3PX_4 and if F, G, O are the centres of these faces and of the eightcell, $FPGO$ is a square and the net (C_8), to which the cell $C_8^{(2)}$ belongs, projects itself on the plane of the square PR as a plane-filling of squares (fig. 7), whilst the intersecting space normal to the diagonal PR of the square projects itself in a normal to that line. We expressed this in the second communication by saying that the problem of the section of a four-dimensional polytope by a threedimensional space has lost here two of its dimensions. If now amongst the lines normal to the diagonal PR line a passes through R, line b passes through the points S_1, T_1 on the sides RS, RT for which $RS_1 = \frac{1}{4} RS$, $RT_1 = \frac{1}{4} RT$, and line c passes through the midpoints S_2, T_2 of RS, RT, then the position a of the intersecting space corresponds to the first and the fifth figure of the fourth column of plate II, the position b corresponds to the second and the fourth figure, the position c corresponds to the third one.

A second remark refers to the position of the sections obtained in the third and the fourth column. The first and the third figure of the third column are equal to the first and the fifth figure of the fourth column; also the middle figures of the two columns are equal. But there is a difference in position. In the figures of the third column the axis MN of period four is vertical, in the figures of the fourth column this axis MN is horizontal. This is not accidental. As both columns represent the sections with the polarly circumscribed $C_8^{(4)}$ and the bodily circumscribed $C_2^{(8)}$ in the same orientation, it proves that the axes MN of the sections of the erect sixteencells and those of any of the two groups of the inclined sixteencells are normal to one another, from which may be derived that the three axes MN of the sections of three sixteencells, any two of which belong to different groups, are normal to one another by twos. We verify this by proving that the axes MN of the sections of two inclined sixteencells of different kinds are normal to one another. Therefore we remark that the limiting spaces $P(X_1X_2X_4)$ and $P(X_2X_3X_4)$ of fig. 6 are parallel to OF — as the line GP parallel to OF lies in $P(X_3X_4)$ — and so the intersecting spaces normal to OF are normal to those limiting spaces of $C_2^{(8)}$. As the sixteencells inscribed in $C_8^{(2)}$ and in an adjacent $C_8^{(2)}$ are one another's mirror-image with respect to the limiting space common to the two eightcells, the sections of both the sixteencells with the intersecting space normal to that limiting space are one another's mirror-image with respect to the plane of intersection of intersecting space and limiting space, i.e. with respect to one of the vertical faces of the rectangular square prism that forms the section of $C_8^{(2)}$. As the axis MN of the figures of column four forms an angle of 45° with each of these four faces, it will be normal to its mirror-image.

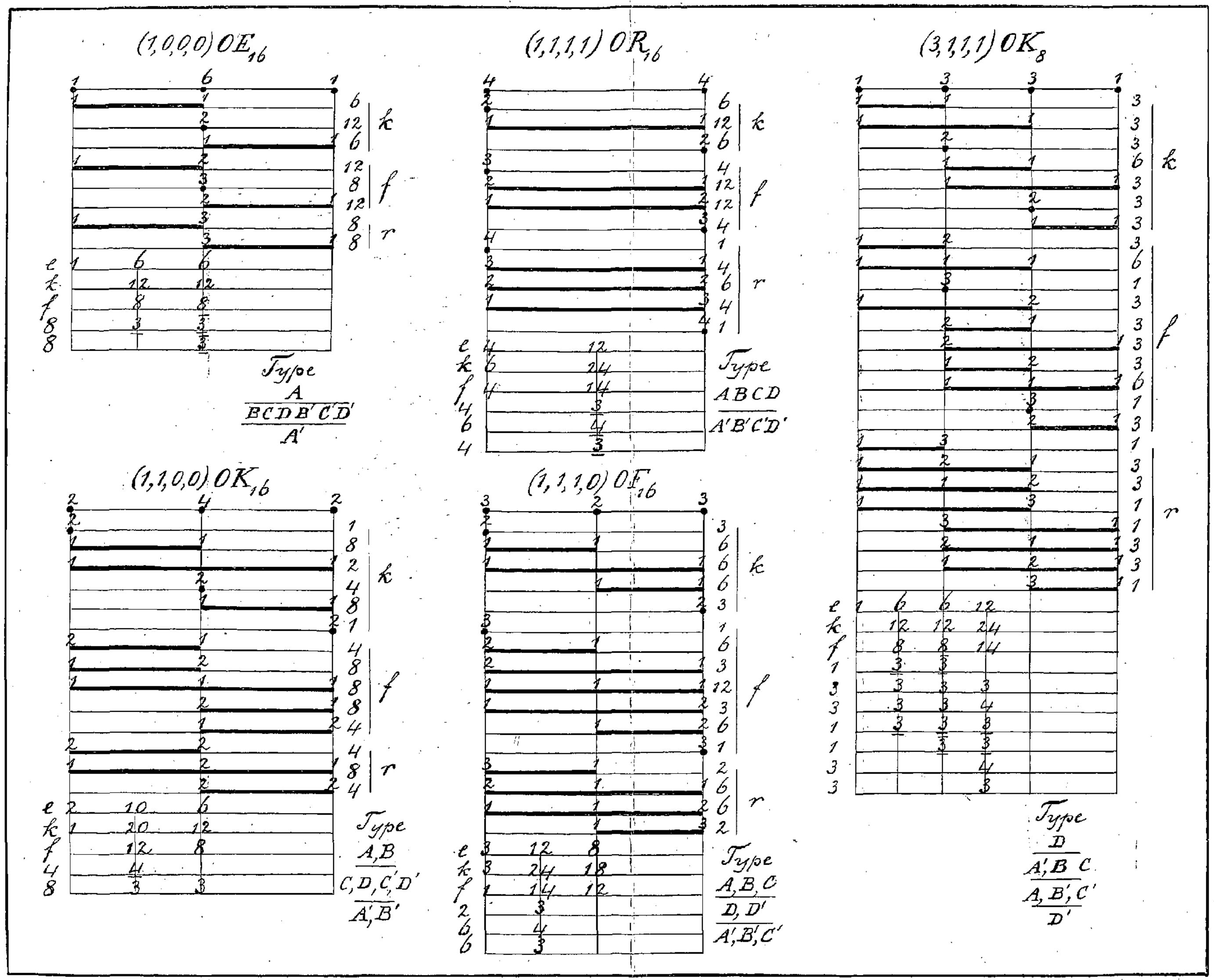
In the first of the three cases — that of the first and the third figure of the third column — only one polyhedron appears, the square double-pyramid, the base of which is a square with side $2\sqrt{2}$, the height of which also is $2\sqrt{2}$. In the following way it is easily proved that this not entirely regular octahedron can form a space-filling by itself Let us consider a net of cubes with edge $2\sqrt{2}$ and divide each of these cubes into six equal square pyramids admitting as base one of the faces of that cube and as common vertex the centre of the cube; then the required net is obtained if we join together to a double-pyramid each pair of pyramids standing on the same base; according to the directions of the axes with period four of these pyramids this net consists of three equally strongly developed groups of polyhedra. We remark that the regular octahedron cannot fill space, but that we obtain a polyhedron that does fill space, as has just been proved, by compressing the regular octahedron in such a manner that the distances of the points of the surface from a plane through four of the six vertices are diminished to $\frac{1}{2}\sqrt{2}$ times the original value.

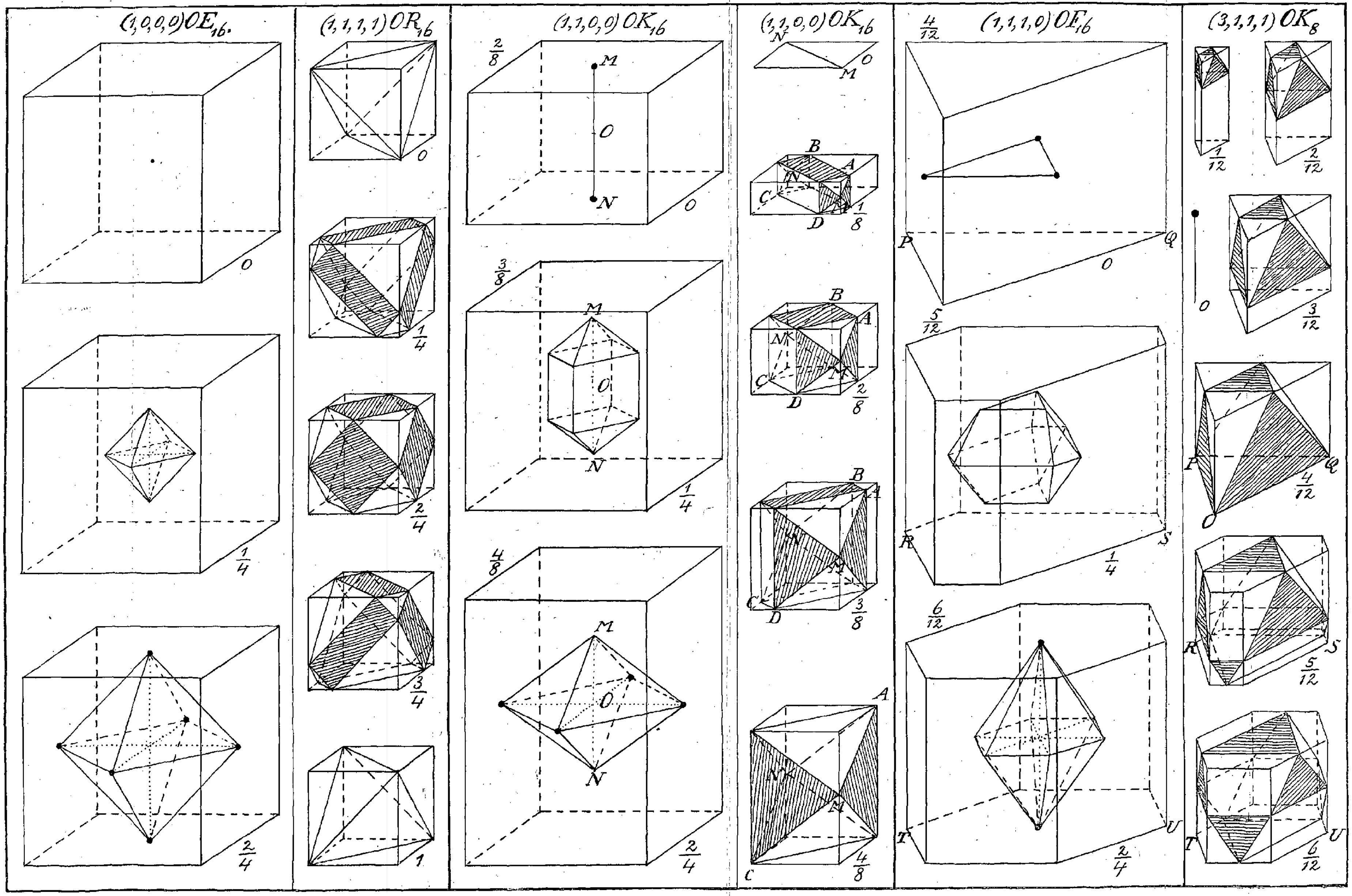
In the third of the three cases — i.e. in that of the middle figure of the five sections of column four — we have again to deal with only one polyhedron, viz. a cube with edge $\sqrt{2}$ bearing on two opposite faces a square pyramid with height $\frac{1}{2}\sqrt{2}$. We show easily that this body has the space-filling property as follows. Let us start from a net of cubes with edge $\sqrt{2}$ and suppose the centre of one of these cubes to be the origin of a rectangular system of coordinates the axes of which are parallel to the edges of the cube. Then let us divide into six equal square pyramids each cube the centre of which has for coordinates either only even or only odd multiples of $\sqrt{2}$, and join each of these pyramids to the adjacent cube; then the required space-filling is obtained. Of these the two figures 8^a and 8^b show the sections with planes $u=2k\sqrt{2}$ and $u=(2k+1)\sqrt{2}$, where u stands for any of the three coordinates; here have been indicated the fibres of the combination (10, 20, 12) running in the direction of the axis with period four, whilst sections normal to tha axis are characterized by yearrings and medullary rays.

If we wish to treat in an analogous manner the second case — i.e. that of the second and the fourth figure of column four — we can start from a net of cubes with edge $2\sqrt{2}$. If we transform this net by assuming inside these cubes concentric and homothetic cubes with edge $\frac{3}{2}\sqrt{2}$ we find, by omitting the boundaries of the original cubes and producing the boundaries of the new ones, a mixed space-filling by cubes and rectangular parallelopipeda characterized by the triplets of edges (1, 1, 1), (3, 3, 3) and (1, 1, 3), (1, 3, 3) with $\frac{1}{2}\sqrt{2}$ as unit. By splitting up each of the cubes of the two sizes into six equal pyramids and joining these pyramids to the adjacent parallelopipeda we get the required space-filling. Here for clearness' sake the figures 9^a and 9^b show the sections corresponding to those of 8^a and 8^b.

In general the space-filling presenting itself here consists of two different polyhedra occurring in three different orientations; in two particular cases one finds however only one polyhedron occurring in three different positions.

Space-fillings normal to OK_8. — Here the problem of the determination of the section of the net (C_8) loses one dimension only; so the consideration of the section of a threedimensional net of cubes by a plane normal to a diagonal shows that three sections always go together whose characteristic fractions differ by $\frac{1}{3}$ from one another. Moreover we have still to bear in mind two things. First we have to observe that the three sections of $C_8^{(2)}$ corresponding to the fractions a, $a+\frac{1}{3}$, $a+\frac{2}{3}$ do not always give three sections of a bodily inscribed sixteencell. If we assume for simplicity a to be situated between the limits 0 and $\frac{1}{3}$ we shall find three sections or $C_{16}^{(2\sqrt{2})}$ if a lies between $\frac{1}{12}$ and $\frac{3}{12}$, i.e. in half the possible cases. In the second place we have to remember that each of two or three sections of sixteencells occurs in two different orientations, being one another's mirror-image with respect to the middle plane of the prism, section of the bodily circumscribed $C_8^{(2)}$. If the intersecting space is normal to the line connecting the centre O of the chosen eightcell with the midpoint K_8 of its edge PQ, then we have only to con-


(1,0,0,0) OE16
Type
A
BCDB'C'D'
A'
(1,1,1,1) OR16
Type
ABCD
A'B'C'D'
(3,1,1,1) OK8
Type
D
A',B C
A,B',C'
D'
(1,1,0,0) OK16
Type
A,B
C,D,C',D'
A',B'
(1,1,1,0) OF16
Type
A,B,C
D,D'
A',B',C'


(1,0,0,0)OE16.
(1,1,1,1)OR16
(1,1,0,0)OK16
(1,1,0,0)OK16
(1,1,1,0)OF16
(3,1,1,1)OK8

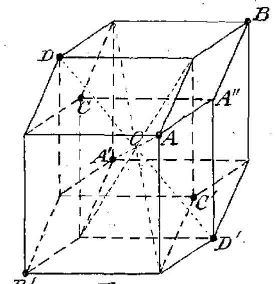


Fig. 1.

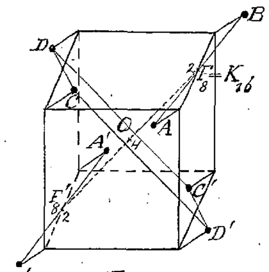


Fig. 2.

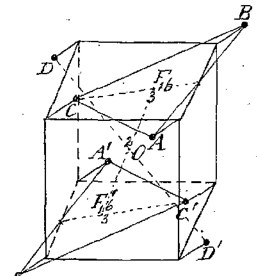


Fig. 3.

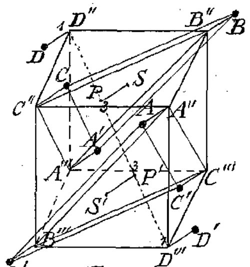


Fig. 4.

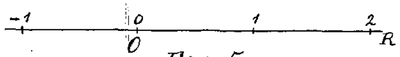


Fig. 5.

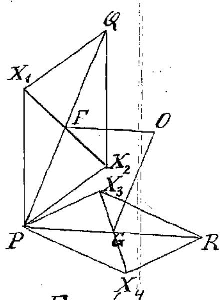


Fig. 6.

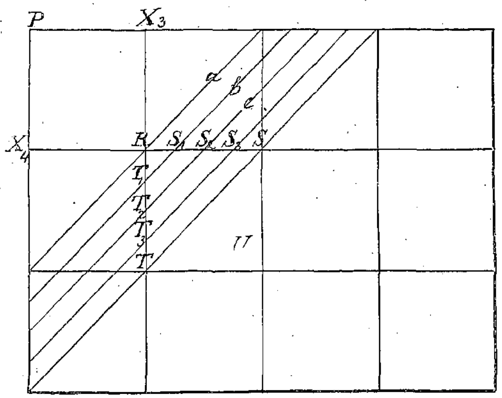


Fig. 7.

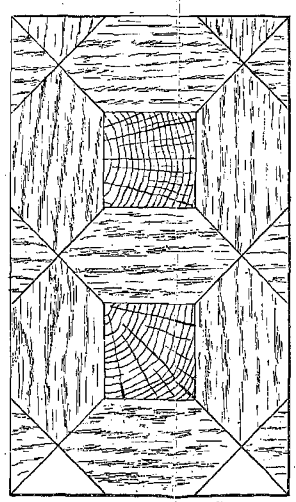


Fig. 8.a

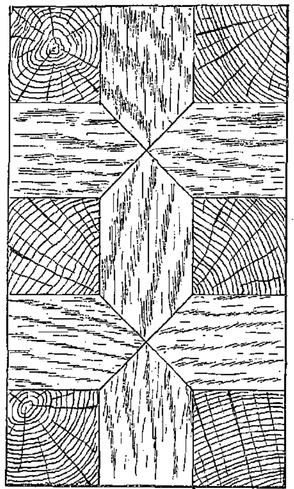


Fig. 8.b

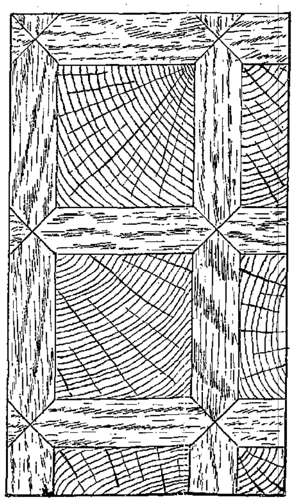


Fig. 9.a

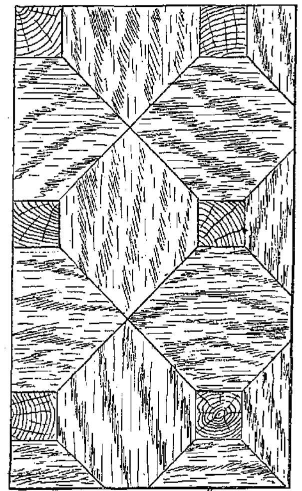


Fig. 9.b

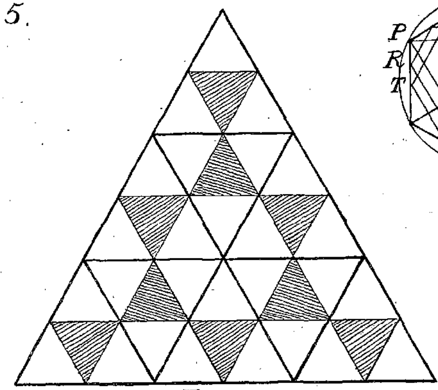


Fig. 10.

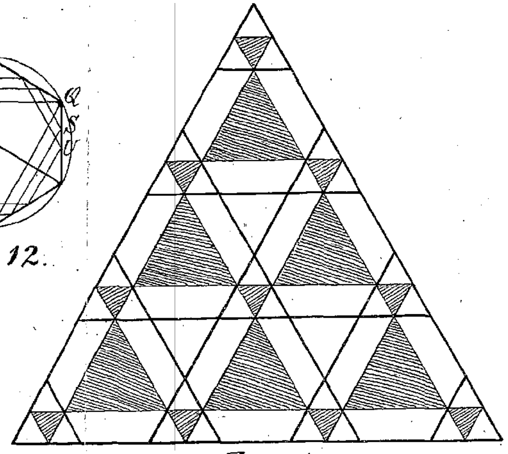


Fig. 11.

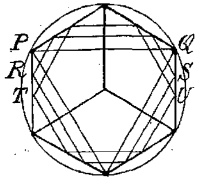


Fig. 12.

sider two bodily adjacent eightcells of the net, the line joining the centres of which is equipollent to PQ, in order to see that these eightcells are intersected in two congruent prisms with common base, whilst the sections with the inscribed inclined sixteencells are symmetric figures with respect to that base. If we consider difference in form only we have to deal therefore with two or three, if we also take into account difference in orientation we have to deal with four or six polyhedra.

If we deprive the problem of the intersection of the system of cells $C_8^{(4)}$ partially penetrating one another, polarly circumscribed to the erect sixteencells, of the one superfluous dimension, we find a system of cubes with edge four, the centres of which are the vertices of a net of cubes with edge two, whilst the edges are parallel to those of the cubes of the net. This system is then to be intersected by a plane normal to a diagonal. By means of a simple diagram we then find that to the three sections $a, a+\frac{1}{3}, a+\frac{2}{3}$ of the net $C_8^{(2)}$ correspond the six sections $\frac{a}{2}, \frac{a}{2}+\frac{1}{6}, \ldots, \frac{a}{2}+\frac{5}{6}$ of the system $C_8^{(4)}$. But of these six different sections of $C_8^{(4)}$ only two give rise to sections of erect sixteencells, viz. those the fractions of which lie between $\frac{2}{6}$ and $\frac{4}{6}$. So we get as the two most regular of the space-fillings presenting themselves here the two indicated by figures in heavy type in the following scheme:

In $C_8^{(4)}$ . . . $0, \frac{1}{6}, \frac{2}{6}, \frac{\mathbf{3}}{\mathbf{6}}, \frac{4}{6}, \frac{5}{6}$	In $C_8^{(4)}$ . . . $\frac{1}{12}, \frac{3}{12}, \frac{\mathbf{5}}{\mathbf{12}}, \frac{\mathbf{7}}{\mathbf{12}}, \frac{9}{12}, \frac{11}{12}$
In $C_8^{(2)}$ . . . $0, \frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{2}}{\mathbf{3}}$	In $C_8^{(2)}$ . . . $\frac{\mathbf{1}}{\mathbf{6}}, \frac{\mathbf{3}}{\mathbf{6}}, \frac{\mathbf{5}}{\mathbf{6}}$

Of these space-fillings the first consists of regular hexagonal double-pyramids (last figure of column five of plate II) and as to shape only one other form, an irregular octahedron (the octahedron of the sixth column with OPQ as base), whilst the second one is built up by three different bodies in that supposition. The diagrams 10 and 11 represent a projection of both on the base of the prisms forming the sections of the including eightcells.

From fig. 12, added partly to fill the page, which shows the sections of a cube by planes normal to an interior diagonal, can be deduced finally that the segments of lines PQ, RS, TU of the three figures of column five — and of the last three figures of column six — of plate II have the same length.

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