Citation:

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as *Melandrium album* (*Lychnis vespertina*), *Hyoscyamus niger*, *Galanthus nivalis*, many *Papilionaceae* and *Epilobium angustifolium*.

In the second place I think it may be useful to refer briefly to the so-called nectarless plants, because it might be argued that these do not support the truth or general validity of the hypothesis, put forward above.

I have already had an opportunity of pointing out, that some plants, which do not contain nectar, have their ovarian-wall covered with *wax*, and others with *glands secreting mucilage*; to these secretions the same biological significance is attached as that, which I think should be attributed to nectar-secretion. Furthermore, I have already mentioned a number of plants, which are recorded as nectarless, but which, nevertheless, must certainly be reckoned among those containing nectar, namely species of *Anemone*, *Clematis*, *Pulsatilla*, and *Paeonia* in the order of *Ranunculaceae*, also *Helianthemum vulgare* and the various species of *Verbascum* and *Hibiscus*. I will only add, that it can be easily shown by chemical means, that the so-called nectarless *Rosaceae*: *Rosa*, *Poterium*, *Agrimonia*, *Aruncus* and *Spiraea* have been wrongly included in this class. Here indeed the nectar is often difficult to observe, but it is none the less present, as in other *Rosaceae*. If the flowers are extracted with water, so that the nectar, which has been thickened by evaporation, passes into solution, the presence of glucose may readily be demonstrated in all these plants. Finally it may be pointed out in this connexion, that very many plants do not require a special protection by nectar, either because the ovary continues its growth without interruption, (on account of early fertilisation, which often already takes place in the bud) or because it is not exposed to the air during the flowering period.

The latter case occurs especially in the genera *Plantago* and *Luzula*, in *Nymphaea alba* and *Erythraea Centaureum*, in *Iuncus*, in most Grasses and in other anemophilous plants.

Mathematics. — "*On a theorem of* PAINLEVÉ's." By Prof. W. KAPTEYN.

1. PAINLEVÉ, in his well-known memoirs on differential equations of the first order, investigated the question when the integrals possess a definite number of values or branches if the independent variable turns round the critical parametric (not the fixed) points.

For differential equations of the first degree

$$\frac{dy}{dx}=\frac{P(x,y)}{Q(x,y)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P and Q represent polynomials in y, he has proved that if the integrals possess n branches, there always exists a substitution

$$u = \frac{y^n + L_{n-1}\, y^{n-1} + \ldots + L_1 y}{M_{n-1}\, y^{n-1} + \ldots + M_1 y + 1} \quad \ldots \quad (2)$$

by which the equation (1) may be reduced to an equation of RICCATI

$$\frac{du}{dx} = Gu^2 + Hu + K \quad \ldots \ldots \ldots \quad (3)$$

the coefficients L, M, G, H, K being functions of x.

Our object in this paper is to prove this proposition in another way, starting from the form of the integral

$$C = \frac{\lambda_n\, y^n + \lambda_{n-1}\, y^{n-1} + \ldots + \lambda_1\, y + \lambda_0}{y^n + \mu_{n-1}\, y^{n-1} + \ldots + \mu_1\, y + \mu_0} \quad \ldots \quad (4)$$

where C represents an arbitrary constant and λ and μ functions of x. The treatment of the two cases $n = 2$ and $n = 3$ will be sufficient to show that the proposition holds good generally.

2. If $n = 2$, it is evident from the integral

$$C = \frac{\lambda_2 y^2 + \lambda_1 y + \lambda_0}{y^2 + \mu_1 y + \mu_0} = const. \quad \ldots \ldots \quad (5)$$

that the differential equation must be of the form

$$\frac{dy}{dx} = \frac{a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0}{b_2 y^2 + 2b_1 y + b_0} \quad \ldots \ldots \quad (6)$$

the coefficients a and b representing functions of x.

Differentiating the equation (5), we find between $a, b, \lambda, \mu,$ the following relations θ being an indefinite factor,

$$\left.\begin{aligned}
\theta a_4 &= \lambda_2' \\
\theta a_3 &= \mu_1\lambda_2' + \lambda_1' - \lambda_2\mu_1' \\
\theta a_2 &= \mu_0\lambda_2' + \mu_1\lambda_1' + \lambda_0' - \lambda_1\mu_1' - \lambda_2\mu_0' \\
\theta a_1 &= \mu_0\lambda_1' + \mu_1\lambda_0' - \lambda_0\mu_1' - \lambda_1\mu_0' \\
\theta a_0 &= \mu_0\lambda_0' - \lambda_0\mu_0' \\
\theta b_2 &= \lambda_1 - \mu_1\lambda_2 \\
\theta b_1 &= \lambda_0 - \mu_0\lambda_2 \\
\theta b_0 &= \mu_1\lambda_0 - \mu_0\lambda_1 .
\end{aligned}\right\} \quad \ldots \quad (7)$$

From the three latter equations (7) may be induced

$$\left.\begin{array}{r} b_0\lambda_2 - b_1\lambda_1 + b_2\lambda_0 = 0 \\ b_1\mu_1 - b_2\mu_0 - b_0 = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and from the five preceding

$$\mu_0' = \theta \begin{vmatrix} a_4 & 1 & 0 & 0 & 0 \\ a_3 & \mu_1 & 1 & 0 & -\lambda_2 \\ a_2 & \mu_0 & \mu_1 & 1 & -\lambda_1 \\ a_1 & 0 & \mu_0 & \mu_1 & -\lambda_0 \\ a_0 & 0 & 0 & \mu_0 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu_1 & 1 & 0 & -\lambda_2 & 0 \\ \mu_0 & \mu_1 & 1 & -\lambda_1 & -\lambda_2 \\ 0 & \mu_0 & \mu_1 & -\lambda_0 & -\lambda_1 \\ 0 & 0 & \mu_0 & 0 & -\lambda_0 \end{vmatrix} .$$

This equation may be easily reduced to an equation of RICCATI. For adding up, in the first determinant the third column multiplied by λ_2 to the fifth and in the second determinant the second and third columns each multiplied by λ_2 to the fourth and last, we get

$$\mu_0' = - \begin{vmatrix} a_4 & 1 & 0 & 0 & 0 \\ a_3 & \mu_1 & 1 & 0 & 0 \\ a_2 & \mu_0 & \mu_1 & 1 & b_2 \\ a_1 & 0 & \mu_0 & \mu_1 & b_1 \\ a_0 & 0 & 0 & \mu_0 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu_1 & 1 & 0 & 0 & 0 \\ \mu_0 & \mu_1 & 1 & b_2 & 0 \\ 0 & \mu_0 & \mu_1 & b_1 & b_2 \\ 0 & 0 & \mu_0 & 0 & b_1 \end{vmatrix} .$$

If now we substitute

$$\mu_1 = \frac{b_0 + b_2\mu_0}{b_1} .$$

in the denominator, and subtract the fourth and fifth columns each multiplied by $\frac{\mu_0}{b_1}$ from the second and third, we find

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu_1 & 1 & 0 & 0 & 0 \\ \mu_0 & \frac{b_0}{b_1} & 1 & b_2 & 0 \\ 0 & 0 & \frac{b_0}{b_1} & b_1 & b_2 \\ 0 & 0 & 0 & 0 & b_1 \end{vmatrix} = b_1^2 - b_0 b_2$$

If we in the same way subtract the fifth column multiplied by $\frac{\mu_0}{b_1}$ from the third, the numerator takes the form

$$\begin{vmatrix} a_4 & 1 & 0 & 0 & 0 \\ a_3 & \frac{b_0+b_2\mu_0}{b_1} & 1 & 0 & 0 \\ a_2 & \mu_0 & \frac{b_0}{b_1} & 1 & b_2 \\ a_1 & 0 & 0 & \frac{b_0+b_2\mu_0}{b_1} & b_1 \\ a_0 & 0 & 0 & \mu_0 & 0 \end{vmatrix} = A\mu_0^2 + B\mu_0 + C$$

where the coefficients have to be determined still.

If we put $\mu_0 = 0$, the coefficient C is found to be

$$C = a_0 \begin{vmatrix} 1 & b_2 \\ \frac{b_0}{b_1} & b_1 \end{vmatrix} = \frac{a_0}{b_1}(b_1^2 - b_0 b_2).$$

Dividing further both members by μ_0^2 and supposing afterwards $\mu_0 = \infty$, we get

$$A = \begin{vmatrix} a_4 & 0 & 0 & 0 & 0 \\ a_3 & \frac{b_2}{b_1} & 1 & 0 & 0 \\ a_2 & 1 & \frac{b_0}{b_1} & 1 & b_2 \\ a_1 & 0 & 0 & \frac{b_2}{b_1} & b_1 \\ a_0 & 0 & 0 & 1 & 1 \end{vmatrix} = - a_4 b_1 \begin{vmatrix} \frac{b_2}{b_1} & 1 \\ 1 & \frac{b_0}{b_1} \end{vmatrix} = \frac{a_4}{b_1}(b_1^2 - b_0 b_2).$$

Differentiating both members with respect to μ_0, and substituting $\mu_0 = 0$, we get for B the form

$$B = \begin{vmatrix} a_4 & 0 & 0 & 0 & 0 \\ a_3 & \frac{b_2}{b_1} & 1 & 0 & 0 \\ a_2 & 1 & \frac{b_0}{b_1} & 1 & b_2 \\ a_1 & 0 & 0 & \frac{b_0}{b_1} & b_1 \\ a_0 & 0 & 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_4 & 1 & 0 & 0 & 0 \\ a_3 & \frac{b_0}{b_1} & 1 & 0 & 0 \\ a_2 & 0 & \frac{b_0}{b_1} & 0 & b_2 \\ a_1 & 0 & 0 & \frac{b_2}{b_1} & b_1 \\ a_0 & 0 & 0 & 1 & 0 \end{vmatrix}.$$

The first of these determinants is identically zero; the second developed, gives

$$B = -\frac{b_0^2}{b_1} a_4 + b_0 a_3 - b_1 a_2 + b_2 a_1 - \frac{b_2^2}{b_1} a_0.$$

Hence μ_0 satisfies the following equation of RICCATI

$$\mu_0' = -\frac{a_4}{b_1}(b_1^2 - b_0 b_2)\mu_0^2 + \frac{1}{b_1}(b_0^2 a_4 - b_0 b_1 a_3 + b_1^2 a_2 - b_1 b_2 a_1 + b_2^2 a_0)\mu_0 -$$

$$-\frac{a_0}{b_1}(b_1^2 - b_0 b_2). \quad \ldots \ldots \ldots \quad (9)$$

We now proceed to find the substitution of PAINLEVÉ.

From the general integral

$$C = \frac{\lambda_2 y^2 + \lambda_1 y + \lambda_0}{y^2 + \mu_1 y + \mu_0}$$

it is evident that μ_0 is that particular solution of the equation (9) which satisfies the equation

$$y^2 + \frac{b_0 + b_2 \mu_0}{b_1} y + \mu_0 = 0$$

if we attribute to y that particular integral of (6) which corresponds to the value $C = \infty$.

Therefore

$$\mu_0 = -\frac{b_1 y^2 + b_0 y}{b_2 y + b_1}$$

is the substitution which reduces the differential equation (6) to (9).

3. From the preceding we may also deduce the conditions which must be satisfied by the given differential equation. For the three last equations (7) give

$$\frac{d}{dx}\left(\frac{b_2}{b_1}\right) = \frac{d}{dx}\,\frac{\lambda_1 - \mu_1 \lambda_2}{\lambda_0 - \mu_0 \lambda_2} \text{ and } \frac{d}{dx}\left(\frac{b_0}{b_1}\right) = \frac{d}{dx}\,\frac{\mu_1 \lambda_0 - \mu_0 \lambda_1}{\lambda_0 - \mu_0 \lambda_2}$$

or

$$\theta\,(b_1 b_2' - b_2 b_1') = -\,b_0 \lambda_2' + b_1 \lambda_1' - b_2 \lambda_0' - b_1 \lambda_2 \mu_1' + b_2 \lambda_2 \mu_0'$$

and

$$\theta\,(b_1 b_0' - b_0 b_1') = b_0 \mu_0 \lambda_2' - b_1 \mu_0 \lambda_1' + b_2 \mu_0 \lambda_0' + b_1 \lambda_0 \mu_1' - b_2 \lambda_0 \mu_0'.$$

Combining each of these with the five first equations (7) and eliminating $\lambda_2'\ \lambda_1'\ \lambda_0'\ \mu_1'\ \mu_0'$ we may write the conditions

$$\begin{vmatrix} a_4 & 1 & 0 & 0 & 0 & 0 \\ a_3 & \mu_1 & 1 & 0 & -\lambda_2 & 0 \\ a_2 & \mu_0 & \mu_1 & 1 & -\lambda_1 & -\lambda_2 \\ a_1 & 0 & \mu_0 & \mu_1 & -\lambda_0 & -\lambda_1 \\ a_0 & 0 & 0 & \mu_0 & 0 & -\lambda_0 \\ (b_1 b_2', & -b_0 & b_1 & -b_2 & -b_1 \lambda_2 & b_2 \lambda_2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} a_4 & 1 & 0 & 0 & 0 & 0 \\ a_3 & \mu_1 & 1 & 0 & -\lambda_2 & 0 \\ a_2 & \mu_0 & \mu_1 & 1 & -\lambda_1 & -\lambda_2 \\ a_1 & 0 & \mu_0 & \mu_1 & -\lambda_0 & -\lambda_1 \\ a_0 & 0 & 0 & \mu_0 & 0 & -\lambda_0 \\ (b_1b_0') & b_0\mu_0 & -b_1\mu_0 & b_2\mu_0 & b_1\lambda_0 & -b_2\lambda_0 \end{vmatrix} = 0 .$$

where (b_1b_2') and (b_1b_0') mean $b_1b_2' - b_2b_1'$ and $b_1b_0' - b_0b_1'$ respectively.

Reducing these determinants in the same way as before, we have immediately

$$\begin{vmatrix} a_4 & 1 & 0 & 0 & 0 & 0 \\ a_3 & \mu_1 & 1 & 0 & 0 & 0 \\ a_2 & \mu_0 & \frac{b_0}{b_1} & 1 & b_2 & 0 \\ a_1 & 0 & 0 & \frac{b_0}{b_1} & b_1 & b_2 \\ a_0 & 0 & 0 & 0 & 0 & b_1 \\ \alpha & \beta & \gamma & \delta & \varepsilon & \zeta \end{vmatrix} = 0 \quad \ldots\ldots \quad (10)$$

the latter row representing the following values

$$\left.\begin{matrix} \alpha = (b_1b_2') & \beta = -b_0 & \gamma = b_1 & \delta = -b_2 & \varepsilon = 0 & \zeta = 0 \\ \alpha = (b_1b_0') & \beta = b_0\mu_0 & \gamma = 0 & \delta = 0 & \varepsilon = -b_1^2 & \zeta = b_1b_2 \end{matrix}\right\} \quad . \quad . \quad (11)$$

If we write $\mu_1 = \frac{b_0 + b_2\mu_0}{b_1}$ the determinant (10) takes the form $A\mu_0 + B$. By differentiation with respect to μ_0, A is determined by

$$A = -a_4b_1 \begin{vmatrix} \frac{d\beta}{d\mu_0} & \gamma & \delta & \varepsilon \\ \frac{b_2}{b_1} & 1 & 0 & 0 \\ 1 & \frac{b_0}{b_1} & 1 & b_2 \\ 0 & 0 & \frac{b_0}{b_1} & b_1 \end{vmatrix}$$

or

$$A = -a_4\, b_1\, (b_1^2 - b_0\, b_2) \left[\frac{1}{b_1} \frac{d\beta}{d\mu_0} - \frac{b_2}{b_1^2}\gamma - \frac{1}{b_1}\delta + \frac{b_0}{b_1^3}\varepsilon \right].$$

In both cases this expression vanishes. Therefore both conditions are found by writing $\mu_0 = 0$ in the equation (10). In this way the conditions we looked for, are the following

$$\begin{vmatrix} a_4 & 1 & 0 & 0 & 0 & 0 \\ a_3 & \frac{b_0}{b_1} & 1 & 0 & 0 & 0 \\ a_2 & 0 & \frac{b_0}{b_1} & 1 & b_2 & 0 \\ a_1 & 0 & 0 & \frac{b_0}{b_1} & b_1 & b_2 \\ a_0 & 0 & 0 & 0 & 0 & b_1 \\ \alpha & (\beta)_{\mu_0=0} & \gamma & \delta & \varepsilon & \zeta \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where the last row is given by the relations (11).

4. When $n = 3$, the general integral

$$C = \frac{\lambda_3 y^3 + \lambda_2 y^2 + \lambda_1 y + \lambda_0}{y^3 + \mu_2 y^2 + \mu_1 y + \lambda_0} = \text{const.}$$

shows, that the differential equation must be of the form

$$\frac{dy}{dx} = \frac{a_6 y^6 + a_5 y^5 + a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0}{b_4 y^4 + 4b_3 y^3 + 6b_2 y^2 + 4b_1 y + b_0} \quad . \quad . \quad . \quad (14)$$

with the following relations between the coefficients a, b, λ, μ:

$$\left.\begin{aligned}
\theta a_6 &= \lambda_3' \\
\theta a_5 &= \mu_2\lambda_3' + \lambda_2' - \lambda_3\mu_2' \\
\theta a_4 &= \mu_1\lambda_3' + \mu_2\lambda_2' + \lambda_1' - \lambda_2\mu_2' - \lambda_3\mu_1' \\
\theta a_3 &= \mu_0\lambda_3' + \mu_1\lambda_2' + \mu_2\lambda_1' + \lambda_0' - \lambda_1\mu_2' - \lambda_2\mu_1' - \lambda_3\mu_0' \\
\theta a_2 &= \mu_0\lambda_2' + \mu_1\lambda_1' + \mu_2\lambda_0' - \lambda_0\mu_2' - \lambda_1\mu_1' - \lambda_2\mu_0' \\
\theta a_1 &= \mu_0\lambda_1' + \mu_1\lambda_0' - \lambda_0\mu_1' - \lambda_1\mu_0' \\
\theta a_0 &= \mu_0\lambda_0' - \lambda_0\mu_0' \\
\theta b_4 &= \lambda_2 - \lambda_3\mu_2 \\
4\theta b_3 &= 2\lambda_1 - 2\lambda_3\mu_1 \\
6\theta b_2 &= 3\lambda_0 + \lambda_1\mu_2 - \lambda_2\mu_1 - 3\lambda_3\mu_0 \\
4\theta b_1 &= 2\lambda_0\mu_2 - 2\lambda_2\mu_0 \\
\theta b_0 &= \lambda_0\mu_1 - \lambda_1\mu_0
\end{aligned}\right\} \quad (15)$$

Eliminating alternately the μ's and λ's from the five last equations (15) we have

$$\left.\begin{aligned} &3b_0\lambda_3^2 - 6b_2\lambda_1\lambda_3 + 2b_3(\lambda_1\lambda_2 + 3\lambda_0\lambda_3) - b_4\lambda_1^2 = 0 \\ &6b_1\lambda_3^2 - 6b_2\lambda_2\lambda_3 + 2b_3\lambda_2^2 + b_4(3\lambda_0\lambda_3 - \lambda_1\lambda_2) = 0 \\ &(3\mu_0 - \mu_1\mu_2)\,b_4 + 2\mu_2^2 b_3 - 6\mu_2 b_2 + 6b_1 = 0 \\ &\mu_1^2 b_4 - 2(3\mu_0 + \mu_1\mu_2)\,b_3 + 6\mu_1 b_2 - 3b_0 = 0 \end{aligned}\right\} \quad . \quad . \quad (16)$$

The two latter equations (16) enable us to express μ_2 and μ_1 in function of μ_0. For multiplying the first of these by $2b_3$, the second by b_4, and adding up, we find the following quadratic equation

$$(\mu_1 b_4 - 2\mu_2 b_3)^2 + 6b_2(\mu_1 b_4 - 2\mu_2 b_3) + 3(4b_1 b_3 - b_0 b_4) = 0$$

so

$$\mu_1 b_4 - 2\mu_2 b_3 = -3b_2 + \sqrt{3i_2}$$

where the square root stands for both values, and i_2 represents the expression

$$i_2 = 3b_2^2 - 4b_1 b_3 + b_0 b_4.$$

This result, in connexion with

$$\mu_1(\mu_1 b_4 - 2\mu_2 b_3 + 6b_2) = 3b_0 + 6\mu_0 b_3$$

gives

$$\mu_1 = \frac{3b_0 + 6\mu_0 b_3}{3b_2 + \sqrt{3i_2}} \qquad \mu_2 = \frac{6b_1 + 3\mu_0 b_4}{3b_2 + \sqrt{3i_2}}.$$

Now the first seven equations (15) lead up to

$$\mu_0' = \theta \begin{vmatrix} a_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_5 & \mu_2 & 1 & 0 & 0 & -\lambda_3 & 0 \\ a_4 & \mu_1 & \mu_2 & 1 & 0 & -\lambda_2 & -\lambda_3 \\ a_3 & \mu_0 & \mu_1 & \mu_2 & 1 & -\lambda_1 & -\lambda_2 \\ a_2 & 0 & \mu_0 & \mu_1 & \mu_2 & -\lambda_0 & -\lambda_1 \\ a_1 & 0 & 0 & \mu_0 & \mu_1 & 0 & -\lambda_0 \\ a_0 & 0 & 0 & 0 & \mu_0 & 0 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 1 & 0 & 0 & -\lambda_3 & 0 & 0 \\ \mu_1 & \mu_2 & 1 & 0 & -\lambda_2 & -\lambda_3 & 0 \\ \mu_0 & \mu_1 & \mu_2 & 1 & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ 0 & \mu_0 & \mu_1 & \mu_2 & -\lambda_0 & -\lambda_1 & -\lambda_2 \\ 0 & 0 & \mu_0 & \mu_1 & 0 & -\lambda_0 & -\lambda_1 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 & -\lambda_0 \end{vmatrix}.$$

which reduces to an equation of RICCATI. For adding up in the numerator λ_3 times the third column to the sixth and λ_3 times the fourth to the seventh, and in the denominator λ_3 times the second, the third, and the fourth columns respectively to the fifth, sixth, and seventh, we find

$$\mu_0' = -\begin{vmatrix} a_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_5 & \mu_2 & 1 & 0 & 0 & 0 & 0 \\ a_4 & \mu_1 & \mu_2 & 1 & 0 & b_4 & 0 \\ a_3 & \mu_0 & \mu_1 & \mu_2 & 1 & 2b_3 & b_4 \\ a_2 & 0 & \mu_0 & \mu_1 & \mu_2 & u & 2b_3 \\ a_1 & 0 & 0 & \mu_0 & \mu_1 & 0 & u \\ a_0 & 0 & 0 & 0 & \mu_0 & 0 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & \mu_2 & 1 & 0 & b_4 & 0 & 0 \\ \mu_0 & \mu_1 & \mu_2 & 1 & 2b_3 & b_4 & 0 \\ 0 & \mu_0 & \mu_1 & \mu_2 & u & 2b_3 & b_4 \\ 0 & 0 & \mu_0 & \mu_1 & 0 & u & 2b_3 \\ 0 & 0 & 0 & \mu_0 & 0 & 0 & u \end{vmatrix}$$

where u is determined by the relation

$$\theta u = \frac{6\theta b_2 + \lambda_1\mu_2 - \lambda_2\mu_1}{3} = \theta\,\frac{6b_2 + \mu_1 b_4 - 2\mu_2 b_3}{3} = \frac{\theta}{3}(3b_2 + \sqrt{3i_2}) = \frac{\theta}{3} m.$$

If we substract in the numerator $\frac{3\mu_0}{m}$ times the sixth and seventh columns from the third and fourth and in the denominator $\frac{3\mu_0}{m}$ times the fifth, sixth, and seventh from the second, third and fourth columns, the value of μ'_0 reduces to

$$\mu_0' = -\begin{vmatrix} a_6 & 1 & 0 & 0 & 0 & 0 & 0 \\ a_5 & \mu_2 & 1 & 0 & 0 & 0 & 0 \\ a_4 & \mu_1 & \frac{6b_1}{m} & 1 & 0 & b_4 & 0 \\ a_3 & \mu_0 & \frac{3b_0}{m} & \frac{6b_1}{m} & 1 & 2b_3 & b_4 \\ a_2 & 0 & 0 & \frac{3b_0}{m} & \mu_2 & \frac{m}{3} & 2b_3 \\ a_1 & 0 & 0 & 0 & \mu_1 & 0 & \frac{m}{3} \\ a_0 & 0 & 0 & 0 & \mu_0 & 0 & 0 \end{vmatrix} : \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & \frac{6b_1}{m} & 1 & 0 & b_4 & 0 & 0 \\ \mu_0 & \frac{3b_0}{m} & \frac{6b_1}{m} & 1 & 2b_3 & b_4 & 0 \\ 0 & 0 & \frac{3b_0}{m} & \frac{6b_1}{m} & \frac{m}{3} & 2b_3 & b_4 \\ 0 & 0 & 0 & \frac{3b_0}{m} & 0 & \frac{m}{3} & 2b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{m}{3} \end{vmatrix}.$$

Here the denominator N is evidently independent of μ_0 and may be written

$$N = \frac{m}{3}\begin{vmatrix} 1 & 0 & b_4 & 0 \\ \frac{6b_1}{m} & 1 & 2b_3 & b_4 \\ \frac{3b_0}{m} & \frac{6b_1}{m} & \frac{m}{3} & 2b_3 \\ 0 & \frac{3b_0}{m} & 0 & \frac{m}{3} \end{vmatrix} = \frac{m^3}{27} - \frac{(2b_0b_4 + 4b_1b_3)\,m}{3} + \\ + 4b_0b_3^2 + 4b_1^2b_4 + \frac{3b_0^2b_4^2 - 12b_0b_1b_3b_4}{m}.$$

This takes a simpler form if we eliminate all the powers of m except the first. The definition of m gives

$$m^3 = (36b_2^2 - 3b_0b_4 - 12b_1b_3)\,m + 18b_0b_2b_4 - 72b_1b_2b_3$$

$$\frac{1}{m} = \frac{3b_2 - \sqrt{3i_2}}{3\,(4b_1b_3 - b_0b_4)}$$

hence

$$\frac{3b_0^2b_4^2 - 12b_0b_1b_3b_4}{m} = -\,b_0b_4\,(6b_2 - m).$$

With these values, and putting

$$i_3 = b_0b_2b_4 + 2b_1b_2b_3 - b_2^3 - b_0b_3^2 - b_1^2b_4$$

we obtain finally

$$N = \frac{4i_2}{9}\,m - \frac{4}{3}\,(3i_3 + b_2i_2) = \frac{4}{9}\,i_2\sqrt{3i_2} - 4i_3 = \frac{4}{9}\,(i_2\sqrt{3i_2} - 9i_3)\,. \quad (18)$$

Introducing now the values of μ_2 and μ_1 in function of μ_0 in the numerator, we may reduce this to $A\mu_0^2 + B\mu_0 + C$, where the coefficients are to be determined still.

If we put $\mu_0 = 0$, C is immediately found

$$C = -\,a_0 \begin{vmatrix} 1 & 0 & b_4 & 0 \\ \frac{6b_1}{m} & 1 & 2b_3 & b_4 \\ \frac{3b_0}{m} & \frac{6b_1}{m} & \frac{m}{3} & 2b_3 \\ 0 & \frac{3b_0}{m} & 0 & \frac{m}{3} \end{vmatrix} = -\frac{3}{m}\,a_0N.$$

If we divide further the second and third columns by μ_0 and substitute afterwards $\mu = \infty$, the equation is reducible to

$$A = -\frac{m}{3}\,a_0 \begin{vmatrix} \frac{3b_4}{m} & 1 & 0 & 0 \\ \frac{6b_3}{m} & \frac{6b_1}{m} & 1 & b_4 \\ 1 & \frac{3b_0}{m} & \frac{6b_1}{m} & 2b_3 \\ 0 & 0 & \frac{3b_0}{m} & \frac{m}{3} \end{vmatrix} = -\,a_0 \begin{vmatrix} b_4 & 1 & 0 & 0 \\ 2b_3 & \frac{6b_1}{m} & 1 & b_4 \\ \frac{m}{3} & \frac{3b_0}{m} & \frac{6b_1}{m} & 2b_3 \\ 0 & 0 & \frac{3b_0}{m} & \frac{m}{3} \end{vmatrix} = -\frac{3}{m}\,a_0N.$$

Differentiating the numerator with respect to μ_0 and putting $\mu_0 = 0$ afterwards, we find the value of B. This value consists of two determinants; the first of these is identically zero, therefore

$$
B = - \begin{vmatrix}
a_6 & 1 & 0 & 0 & 0 & 0 & 0 \\
a_5 & \frac{6b_1}{m} & 1 & 0 & 0 & 0 & 0 \\
a_4 & \frac{3b_0}{m} & \frac{6b_1}{m} & 1 & 0 & b_4 & 0 \\
a_3 & 0 & \frac{3b_0}{m} & \frac{6b_1}{m} & 0 & 2b_3 & b_4 \\
a_2 & 0 & 0 & \frac{3b_0}{m} & \frac{3b_4}{m} & \frac{m}{3} & 2b_3 \\
a_1 & 0 & 0 & 0 & \frac{6b_3}{m} & 0 & \frac{m}{3} \\
a_0 & 0 & 0 & 0 & 1 & 0 & 0
\end{vmatrix}
$$

or

$$
\begin{aligned}
B = & - a_6 \left[2b_0 b_1 + \frac{12b_2 - 2m}{4b_1 b_3 - b_0 b_4} (4b_1^3 + b_0^2 b_3 - 6b_0 b_1 b_2) \right] \\
& - a_5 \cdot \frac{2}{3} (b_0 m + 6b_1^2 - 9b_0 b_2) \\
& - a_4 \cdot \frac{2}{3} (b_1 m - 3b_0 b_3) \\
& - a_3 \cdot \frac{2}{3} (b_2 m - 2b_1 b_3 - b_0 b_4) \\
& - a_2 \cdot \frac{2}{3} (b_3 m - 3b_1 b_4) \\
& - a_1 \cdot \frac{2}{3} (b_4 m + 6b_3^2 - 9b_2 b_4) \\
& - a_0 \left[2b_3 b_4 + \frac{12b_2 - 2m}{4b_1 b_3 - b_0 b_4} (4b_3^3 + b_1 b_4^2 - 6b_2 b_3 b_4) \right].
\end{aligned}
$$

With these values the differential of RICCATI takes the form

$$
\mu_0' = - \frac{3}{m} a_6 \mu_0^2 + \frac{B}{N} \mu_0 - \frac{3}{m} a_0 . \quad . \quad . \quad . \quad . \qquad (19)
$$

and the same reasoning as before shows that if the necessary conditions are satisfied the substitution which reduces the given differential equation (14) to the equation (19) may be inferred from

$$
y^3 + \mu_2 y^2 + \mu_1 y + \mu_0 = 0.
$$

Substituting the values (17) we conclude finally that

$$\mu_0 = -\frac{my^3 + 6b_1y^2 + 3b_0y}{3b_4y^2 + 6b_3y + m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

reduces (14) to (19).

5. To determine in this case the conditions, we differentiate the four values $\frac{b_4}{b_2}$, $\frac{b_3}{b_2}$, $\frac{b_1}{b_2}$, $\frac{b_0}{b_2}$ expressed in λ and μ by (15). This gives

$$\left.\begin{aligned}
6\theta(b_4b_2') &= (6b_2\mu_2-3b_4\mu_0)\lambda_3'-(b_4\mu_1+6b_2)\lambda_2'+b_4\mu_2\lambda_1'+3b_4\lambda_0'+\\
&\quad +(b_4\lambda_1+6b_2\lambda_3)\mu_2'-b_4\lambda_2\mu_1'-3b_4\lambda_3\mu_0'\\
6\theta(b_3b_2') &= (3b_2\mu_1-3b_3\mu_0)\lambda_3'-b_3\mu_1\lambda_2'+(b_3\mu_2-3b_2)\lambda_1'+3b_3\lambda_0'+\\
&\quad +b_3\lambda_1\mu_2'+(3b_2\lambda_3-b_3\lambda_2)\mu_1'-3b_3\lambda_3\mu_0'\\
6\theta(b_1b_2') &= -3b_1\mu_0\lambda_3'+(3b_2\mu_0-b_1\mu_1)\lambda_2'+b_1\mu_2\lambda_1'+(3b_1-3b_2\mu_2)\lambda_0'+\\
&\quad +(b_1\lambda_1-3b_2\lambda_0)\mu_2'-b_1\lambda_2\mu_1'+(3b_2\lambda_2-3b_1\lambda_3)\mu_0'\\
6\theta(b_0b_2') &= -3b_0\mu_0\lambda_3'-b_0\mu_1\lambda_2'+(b_0\mu_2+6b_2\mu_0)\lambda_1'+(3b_0-6b_2\mu_1)\lambda_0'+\\
&\quad +b_0\lambda_1\mu_2'-(b_0\lambda_2+6b_2\lambda_0)\mu_1'+(6b_2\lambda_1-3b_0\lambda_3)\mu_0'
\end{aligned}\right\} . (21)$$

Combining each of these equations with the seven former equations (15) and eliminating the quantities $\lambda_3'\lambda_2'\lambda_1'\lambda_0'\mu_2'\mu_1'\mu_0'$, we obtain

$$\begin{vmatrix}
a_6 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
a_5 & \mu_2 & 1 & 0 & 0 & -\lambda_3 & 0 & 0\\
a_4 & \mu_1 & \mu_2 & 1 & 0 & -\lambda_2 & -\lambda_3 & 0\\
a_3 & \mu_0 & \mu_1 & \mu_2 & 1 & -\lambda_1 & -\lambda_2 & -\lambda_3\\
a_2 & 0 & \mu_0 & \mu_1 & \mu_2 & -\lambda_0 & -\lambda_1 & -\lambda_2\\
a_1 & 0 & 0 & \mu_0 & \mu_1 & 0 & -\lambda_0 & -\lambda_1\\
a_0 & 0 & 0 & 0 & \mu_0 & 0 & 0 & -\lambda_0\\
C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7
\end{vmatrix} = 0$$

where the last row is formed by the coefficients of each of the four equations (21). Hence for the first of these

$$C_0 = 6(b_4b_2'), \qquad C_1 = 6b_2\mu_2 - 3b_4\mu_0, \text{ etc.}$$

If we reduce this determinant in the same way as before, the last row becomes in the first place

$$C_0,\ C_1,\ C_2,\ C_3,\ C_4,\ \frac{\lambda_3C_2+C_5}{\theta},\ \frac{\lambda_3C_3+C_6}{\theta},\ \frac{\lambda_3C_4+C_7}{\theta}$$

and secondly

$$C_0,\ C_1,\ C_2-\frac{3\mu_0}{m}\frac{\lambda_3C_2+C_5}{\theta},\ C_3-\frac{3\mu_0}{m}\frac{\lambda_3C_3+C_6}{\theta},\ C_4-\frac{3\mu_0}{m}\frac{\lambda_3C_4+C_7}{\theta},$$
$$\frac{\lambda_3C_2+C_5}{\theta},\ \frac{\lambda_3C_3+C_6}{\theta},\ \frac{\lambda_3C_4+C_7}{\theta},$$

that is for the four cases successively

$$6\,(b_4b_2'),\ 6\,b_2\mu_2-3b_4\mu_0,\ -6\,b_2-\frac{3b_0b_4}{m},\ \frac{6b_1b_4}{m},\ 3b_4,\ -2b_3b_4,\ b_4^2,\ 0$$

$$6\,(b_3b_2'),\ 3\,b_2\mu_1-3b_3\mu_0,\ -\frac{3b_0b_3}{m},\ \frac{6b_1b_3}{m}-3b_2,\ 3b_3,\ -2b_3^2,\ b_3\,b_4,\ 0$$

$$6\,(b_1b_2'),\ -3\,b_1\mu_0,\ -\frac{3b_0b_1}{m},\ \frac{6b_1^2}{m},\ 3b_1-\frac{18b_1b_2}{m},\ b_2m-2b_1b_3,\ b_1b_4,\ -3b_2b_4$$

$$6\,(b_0b_2'),\ -3\,b_0\mu_0,\ -\frac{3b_0^2}{m},\ \frac{6b_0b_1}{m},\ 3b_0-\frac{18b_0b_2}{m},$$
$$-2b_0b_3,\ b_0b_4+2b_2m,\ -12b_2b_3$$

which may be represented for a moment by $D_0D_1D_2D_3D_4D_5D_6D_7$. After these reductions it is evident that only the second column contains the quantities $\mu_0\,\mu_1\,\mu_2$. Hence, with regard to the relations (17), this determinant may be written in the form $A\mu_0+B$, where the value of A is found by differentiating with respect to μ_0 and B by substituting $\mu_0=0$.

In this way A takes the form of a determinant of the eighth order which immediately leads to the following of the sixth order.

$$A=-a_6u\begin{vmatrix} \frac{3b_4}{m} & 1 & 0 & 0 & 0 & 0 \\ \frac{6b_3}{m} & \frac{6b_1}{m} & 1 & 0 & b_4 & 0 \\ 1 & \frac{3b_0}{m} & \frac{6b_1}{m} & 1 & 2b_3 & b_4 \\ 0 & 0 & \frac{3b_0}{m} & \frac{6b_1}{m} & \frac{m}{3} & 2b_3 \\ 0 & 0 & 0 & \frac{3b_0}{m} & 0 & \frac{m}{3} \\ D_1' & D_2 & D_3 & D_4 & D_5 & D_6 \end{vmatrix}=a_6u\left(\frac{3}{m}\right)^4\begin{vmatrix} \frac{m}{3}D'_1 & \frac{m}{3}D_2 & \frac{m}{3}D_3 & \frac{m}{3}D_4 & D_5 & D_6 \\ b_4 & \frac{m}{3} & 0 & 0 & 0 & 0 \\ 2b_3 & 2b_1 & \frac{m}{3} & 0 & b_4 & 0 \\ \frac{m}{3} & b_0 & 2b_1 & \frac{m}{3} & 2b_3 & b_4 \\ 0 & 0 & b_0 & 2b_1 & \frac{m}{3} & 2b_3 \\ 0 & 0 & 0 & b_0 & 0 & \frac{m}{3} \end{vmatrix}$$

where $D_1'=\frac{dD_1}{d\mu_0}$.

Developing this determinant, and putting

$$\frac{m^4}{81}-\frac{4b_1b_3+2b_0b_4}{9}m^2+\frac{4b_0b_3^2+4b_1^2b_4}{3}m+b_0b_4(b_0b_4-4b_1b_3)=P$$

we have

$$A=a_0u\left(\frac{3}{m}\right)^4P\left[\frac{m^2}{9}D_1'-\frac{b_4m}{3}D_2-\frac{2b_3m}{3}D_3-\frac{m^2}{9}D_4+2b_1D_5+b_0D_6\right].$$

If we introduce now the values of the quantities D in the last factor, this leads in the four different cases to

$$-\frac{2b_4}{3}m^2+4b_2b_4m+2b_4(b_0b_4-4b_1b_3)$$

$$-\frac{2b_3}{3}m^2+4b_2b_3m+2b_3(b_0b_4-4b_1b_3)$$

$$-\frac{2b_1}{3}m^2+4b_1b_2m+2b_1(b_0b_4-4b_1b_3)$$

$$-\frac{2b_0}{3}m^2+4b_0b_2m+2b_0(b_0b_4-4b_1b_3).$$

If we observe that we have by definition

$$b_0b_4-4b_1b_3=\frac{m^2-6b_2m}{3}$$

it is evident that in all cases $A=0$.

The conditions are therefore determined by $B=0$, and this may be written, after a slight reduction

$$\begin{vmatrix}
a_6 & \frac{m}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_5 & 2b_1 & \frac{m}{3} & 0 & 0 & 0 & 0 & 0 \\
a_4 & b_0 & 2b_1 & \frac{m}{3} & 0 & b_4 & 0 & 0 \\
a_3 & 0 & b_0 & 2b_1 & \frac{m}{3} & 2b_3 & b_4 & 0 \\
a_2 & 0 & 0 & b_0 & 2b_1 & \frac{m}{3} & 2b_3 & b_4 \\
a_1 & 0 & 0 & 0 & b_0 & 0 & \frac{m}{3} & 2b_3 \\
a_0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{m}{3} \\
D_0 & \frac{m}{3}(D_1)_0 & \frac{m}{3}D_2 & \frac{m}{3}D_3 & \frac{m}{3}D_4 & D_5 & D_6 & D_7
\end{vmatrix}=0$$

where the elements of the last row are respectively in the four cases :

$$6(b_4 b_2'),\ 12 b_1 b_2,\ -(b_0 b_4 + 2 b_2 m),\ 2 b_1 b_4,\ b_4 m,\ -2 b_3 b_4,\ b_4^2,\ 0$$

$$6(b_3 b_2'),\ 3 b_0 b_2,\ -b_0 b_3,\ 2 b_1 b_3 - b_2 m,\ b_3 m,\ -2 b_3^2,\ b_3 b_4,\ 0$$

$$6(b_1 b_2'),\ 0,\ -b_0 b_1,\ 2 b_1^2,\ b_1(m - 6 b_2),\ b_2 m - 2 b_1 b_3,\ b_1 b_4,\ -3 b_2 b_4$$

$$6(b_0 b_2'),\ 0,\ -b_0^2,\ 2 b_0 b_1,\ b_0(m - 6 b_2),\ -2 b_0 b_3,\ b_0 b_4 + 2 b_2 m,\ -12 b_2 b_3$$

6. Following the same way in the general case, we obtain for μ_0' the quotient of two determinants each of order $2n+1$. If we reduce these as before, the denominator will be seen to be independent of λ and μ; and the numerator will only contain the quantities $\mu_{n-1}, \mu_{n-2} \cdot\cdot \mu_1, \mu_0$ in two columns. Now $\mu_{n-1}, \mu_{n-2}, \cdot\cdot \mu_1$ may be expressed as linear functions of μ_0, and this proves at once that the numerator must be a polynomial of the second degree in μ_0. If, therefore the necessary conditions are satisfied, the quantity μ_0 is an integral of an equation of RICCATI. The substitution which reduces the given differential equation to this equation of RICCATI will then be found from

$$y^n + \mu_{n-1} y^{n-1} + \ldots \mu_1 y + \mu_0 = 0$$

by determining $\mu_{n-1}, \ldots \mu_1$ in function of μ_0 and expressing μ_0 in function of y.

Physics. — "*The law of shift of the central component of a triplet in a magnetic field.*" By Prof. P. ZEEMAN.

In two communications to this Academy [1]) on "Change of wave-length of the middle line of triplets" I gave conclusive evidence obtained by means of MICHELSON's echelon-spectroscope that the central line of some triplets is shifted. The fact of this displacement was established simultaneously with my own observations by GMELIN [2]) and JACK [3]). GMELIN first gave the law of shift in the case of the mercury line 5791. According to him the change of wavelength under consideration is proportional to the square of the magnetic force.

In the second part of a former paper on "Magnetic resolution of spectral lines and magnetic force" measurements concerning the asymmetrical resolution of the mercury line 5791 are given [4]).

1) P. ZEEMAN. These Procedings February 1908, April 1908.

2) GMELIN. Physikalische Zeitschrift. 9. Jahrgang S. 212—214. 1908.

3) JACK see VOIGT. Magneto-optik. S. 178.

4) ZEEMAN. These Proceedings November 1907.