From a comparative ontogenetical point of view, therefore, also the value of the urethra before and behind the fossa navicularis is different. For, whereas behind the fossa navicularis only a very small portion of the wall can be considered as a production of the phallus-frame, perhaps the vertical part of the lumen as it is found in the urethra of man, this changes before the fossa navicularis in such a way that there the greater part of the wall originates from that frame; therefore behind this fossa the urethra is principally homologous to the "Harnurethra", before it to the "Samenurethra".

Mathematics. — "*On bicuspidal curves of order four.*" By Prof. JAN DE VRIES.

1. It is easy to see, that each curve of order four, C_4, with two cusps can be represented by the equation

$$x_1^2 x_2^2 + 2x_1 x_2 x_3^2 + 2b_1 x_1 x_3^3 + 2b_2 x_2 x_3^3 + c x_3^4 = 0.$$

The triangle of reference has then the cusps O_1, O_2 and the point of intersection O_3 of the cuspidal tangents as vertices.

From the equation

$$(x_1 x_2 + x_3^2)^2 + 2(b_1 x_1 + b_2 x_2 + b_3 x_3)\, x_3^3 = 0,$$

where $2b_3 = c - 1$, is evident that

$$b_x \equiv b_1 x_1 + b_2 x_2 + b_3 x_3 = 0$$

represents the *double tangent* d of C_4 and that the conic

$$u \equiv x_1 x_2 + x_3^2 = 0$$

passes through the tangential points D_1, D_2 of d and *osculates* C_4 in the cusps O_1 and O_2.

By combining the equations

$$u^2 + 2b_\lambda x_3^3 = 0 \quad \text{and} \quad u = \lambda b_x x_3$$

we understand that the conics A_2 through O_1, O_2, D_1 and D_2 generate a system of pairs of points on C_4, which are lying in pairs on the rays

$$2x_3 + \lambda^2 b_x = 0$$

of the pencil, having the point of intersection H of $k \equiv O_1 O_2$ and d as vertex.

As this system of points with the curve is given we shall denote it as the *fundamental involution* F_2.

If we put $\lambda^2 = \mu$, it follows from

$$2x_3 + \mu b_x = 0 \quad , \quad u^2 = \mu b_x^2 x_3^2 ,$$

that C_4 can be generated by a pencil of conics $(O_1 O_2 D_1 D_2)$ arranged in the pairs of an involution and a pencil of lines (H) between which such a projective relation exists that the rays d and k through H correspond to the double-elements of the involution, the first of which is composed of the right lines d and k. The locus of the points of intersection of corresponding elements thus consists of the line d and a C_4 with cusps O_1, O_2.

The polar line h of point $H\,(b_2, -b_1, 0)$ with respect to the conic A_2,

$$x_1x_2 + x_3^2 = \lambda\,(b_1x_1 + b_2x_2 + b_3x_3)\,x_3,$$

has as equation $b_2(x_2-\lambda b_1x_3) - b_1(x_1-\lambda b_2x_3) = 0$ or

$$b_1x_1 = b_2x_2.$$

On the line h lie the points Q_1, Q_2, which are connected with the pair of points P_1, P_2 of F_2 generated by A_2 in such a way that we have

$$Q_1 \equiv (O_1P_1,\, O_2P_2) \text{ and } Q_2 \equiv (O_1P_2,\, O_2P_1).$$

The fundamental involution F_2 is thus projected out of O_1 and out of O_2 in the same involutory system of points (Q_1, Q_2). Now Q_1 is the projection of *two* points P_1 and P_1' of C_4, so it is conjugate to two points, Q_2 and Q_2', by means of F_2. Therefore the pairs Q_1, Q_2 form on h an involutory correspondence (2,2).

2. The points of C_4 are projected out of O_1 and O_2 by two pencils in correspondence (2,2); the line k is for both systems a branch-ray, because it is conjugate to the two cuspidal tangents k_1 and k_2; the remaining branch-rays are the tangents out of O_1 and O_2 to C_4.

These tangents are represented by

$$2b_1x_1^3 + 2b_3x_1^2x_3 - 2b_2x_1x_3^2 - b_2^2x_3^3 = 0,$$

$$2b_2x_2^3 + 2b_3x_2^2x_3 - 2b_1x_2x_3^2 - b_1^2x_3^3 = 0.$$

Through the points of intersection of these two three-rays passes the figure, represented by

$$(b_1^2x_1^3-b_2^3x_2^3) + b_3x_3\,(b_1^2x_1^2-b_2^2x_2^2)-b_1b_2x_3^2\,(b_1x_1-b_2x_2) = 0.$$

It is composed of the line h,

$$b_1x_1 = b_2x_2,$$

and the conic

$$(b_1x_1 + b_2x_2)\,bx - b_1b_2\,(x_1x_2 + x_3^2) = 0.$$

The tangents r_1, s_1, t_1 out of O_1 can thus be conjugated to the tangents r_2, s_2, t_2 out of O_2 in such a way that the points of intersection $R \equiv r_1r_2$, $S \equiv s_1s_2$, $T \equiv t_1t_2$ lie with the point of intersection of the cuspidal tangents on a right line h.

At the same time a new proof has been given for the well-known property [1]), according to which the singular elements (branch-elements and double-elements conjugate to them) of a correspondence (2,2) can be arranged in such a way that the singular elements of the first system correspond projectively to those of the second one.

For, if two pencils are connected by a (2, 2) we have but to rotate them around their vertices until a branch-ray of the first pencil coincides with a branch-ray of the second; in the new position they then generate a C_4 with two cusps. From this is evident that there are four projectivities between the singular elements [2]).

The (2 , 2) between the pencils $x_1 = \lambda x_3$ and $x_2 = \mu x_3$ has as equation

$$\lambda^2\mu^2 + 2\lambda\mu + 2b_1\lambda + 2b_2\mu + c = 0.$$

By the points of h these pencils are arranged in the projectivity

$$b_1\lambda = b_2\mu'$$

By eliminating λ we find out of these two relations the equation of the correspondence (2 , 2) between the points which conjugate rays of the pencils (O_1) and (O_2) generate on h. And now it is evident from

$$b^2_2\, \mu^2\, \mu'^2 + 2b_1\, b_2\, \mu\mu' + 2b_1^2\, b_2\, (\mu + \mu') + b_1^2\, c = 0$$

that this correspondence is involutory.

This result is in accordance with the well-known property [3]), according to which a (2 , 2) between two collocal systems is involutory when the two systems have the same branch-elements.

3. Evidently the involutory (2, 2) on h does not differ from the (2, 2) which was deduced from the fundamental involution F_2. Its coincidences arise from the four tangents which one can draw from H to C_4. Indeed, *the polarcurve of H consists of the line h and the conic u* (passing through the points of contact of d).

If the branch-point $R \equiv r_1 r_2$ is conjugate to the double-point R', then R' must be the point of intersection of the rays which the points of contact R_1 and R_2 of r_1 and r_2 project out of O_2 and O_1.

We conclude from this that *the tangential points R_1, S_1 T_1 of the*

[1]) Emil Weyr, *Beiträge zur Curvenlehre*, Vienna 1880, Alfred Hölder, p. 32, or *Annali di Matematica*, 1871, IV, p. 272.

[2]) In my paper "*Over vlakke krommen van de vierde orde met twee dubbelpunten*" (N. Archief voor Wiskunde, 1888, XIV, p. 193) I have applied the properties of the (2,2) correspondence to those curves.

[3]) Emil Weyr „*Ueber einen Correspondenzsatz*", Sitz. ber. der K. Akad. in Wien, 1883, LXXXVII, p. 595, or my paper under the same title in N. Archief voor Wiskunde, 1907, VII, p. 469.

tangents r_1, s_1, t_1 are projected out of the point H into the tangential points R_2, S_2, T_2 of the tangents r_2, s_2, t_2.

If S' corresponds as a double-point of the (2, 2) to $S \equiv s_1 s_2$, then it follows from

$O_1(RR'SS') = O_2(RR'SS')$, that we have $O_1(RR'SS') = O_2(R'RS'S)$.

From this follows that *the points R_1, R_2, S_1, S_2 are connected with O_1, O_2 by a conic.* Also the groups $O_1, O_2, R_1, R_2, T_1, T_2$ and $O_1, O_2, S_1, S_2, T_1, T_2$ lie on conics.

If $K \equiv hk$ we find out of

$$O_1\,(O_3\,KRR') = O_2\,(O_3\,K'RR') = O_2\,(K'\,O_3\,R'R),$$

that through R_1 and R_2 passes a conic which is touched in O_1 and O_2 by the cuspidal tangents. The pairs of points S_1, S_2 and T_1, T_2. procure two analogous conics.

If two arbitrary points X and Y of h are projected out of O_1 and O_2, then the points (O_1X, O_2Y) and (O_1Y, O_2X) lie in a right line through H.

From this follows that H bears three right lines which contain successively the pairs of points

$$\left.\begin{matrix} 1 \equiv r_1 s_2 \\ 4 \equiv r_2 s_1 \end{matrix}\right\} \qquad \left.\begin{matrix} 2 \equiv r_1 t_2 \\ 5 \equiv r_2 t_1 \end{matrix}\right\} \qquad \left.\begin{matrix} 3 \equiv s_1 t_2 \\ 6 \equiv s_2 t_1 \end{matrix}\right\}$$

Above we found that these six points lie on a conic and form two hexagons having O_1 and O_2 as point of Brianchon: it is now evident that they determine a third hexagon, having H as point of Brianchon.

4. From $(kr_1\,s_1\,t_1) = (kr_2\,s_2\,t_2)$ follows

$$(kr_1 s_1 t_1) = (r_2 k t_2 s_2) = (s_2 t_2 k r_2) = (t_2 s_2 r_2 k).$$

So we can bring through O_1 and O_2 three conics $\varrho_2, \sigma_2, \tau_2$ with respect to which the line k has as poles the points R, S, T, whilst containing successively the pairs of points 3,6; 2,5 and 1,4.

On these three conics the pencils (O_1) and (O_2), arranged in (2,2) determine, just as on h, involutory correspondences (2,2); for, the two systems of points generated on them have again the branch-points in common.

If M_1, M_2 is a pair of the (2,2) determined on ϱ_2, then the points (O_1M_1, O_2M_2) and (O_1M_2, O_2M_1) lie on C_4 and in *one* line with the point R, namely on the polar line of the point (M_1M_2, O_1O_2) with respect to ϱ_2.

The pencils with vertices R, S and T generate therefore on C_4 *three more fundamental involutions of pairs of points* where again each ray contains two pairs. They differ from F_2 in this, that unlike the former they do not contain the tangential points of the double tangent as a pair.

For $M_1 \equiv M_2$ we have a coincidence of the (2,2). From this is evident that the tangential points of the four tangents which can still be drawn from R, S or T to C_4, are every time connected with O_1 and O_2 by a conic $(\varrho_2, \sigma_2, \tau_2)$.

In an analogous way as for F_2 we find by paying attention to the singular elements of the (2, 2) on ϱ_2, σ_2 and τ_2, that *the lines* S_1T_2 *and* S_2T_1 *concur in* R, *the lines* R_1T_2 *and* R_2T_1 *in* S *the lines* R_1S_2 *and* R_2S_1 *in* T.

5. The polarcurve of the point $(y_1, y_2, 0)$ has as equation

$$y_1(x_1x_2^2 + x_2x_3^2 + b_1x_3^3) + y_2(x_1^2x_2 + x_1x_3^2 + b_2x_3^3) = 0,$$

or

$$(y_2x_1 + y_1x_2)(x_1x_2 + x_3^2) + (b_1y_1 + b_2y_2)x_3^3 = 0.$$

By combination with the equation

$$(x_1x_2 + x_3^2)^2 + 2b_xx_3^3 = 0$$

of the C_4 is evident that the points of intersection of the two curves lie on $x_1x_2 + x_3^2 = 0$ and on the curve

$$2(y_2x_1 + y_1x_2)b_x = (b_1y_1 + b_2y_2)(x_1x_2 + x_3^2).$$

Therefore the tangential points of the tangents out of a point of O_1O_2 lie on a conic η_2.

For $y_1 : y_2 = b_2 : b_1$, i.e. the point $K \equiv hk$, we find the conic $(b_1x_1 + b_2x_2)b_x = b_1b_2(x_1x_2 + x_3^2)$ through the points 1,2,3,4,5,6.

Out of the equation

$$y_1\{b_1(x_1x_2 + x_3^2) - 2x_2b_x\} + y_2\{b_2(x_1x_2 + x_3^2) - 2x_1b_x\} = 0$$

is evident that the conics η_2 form a pencil having as basis the points of intersection of $x_1x_2 + x_3^2 = 0$ with $b_x = 0$ (the points D_1, D_2) and two points of $b_1x_1 = b_2x_2$ (the line h).

One of the pairs of lines consists of the lines d and h; it contains the tangential points of the tangents out of H, two of which are united in d.

The other two pairs of lines belong to two points of O_1O_2, for which the six tangential points lie every time on two lines passing through D_1 and D_2.

6. If (y_k) is a point of d, thus $b_y = 0$, then its polar curve with respect to C_4 is represented by

$$2(y_1x_2 + y_2x_1 + 2y_3x_3)(x_1x_2 + x_3^2) + 6y_3x_3^2b_x = 0.$$

The points of intersection of these curve with C_4 which are not situated at the same time on $x_1x_2 + x_3^2 = 0$ lie on the conic ξ_2,

$$3y_3(x_1x_3 + x_2^2) = 2(y_2x_1 + y_1x_2 + 2y_3x_3)\,x_3\,.$$

So the tangential points of the four tangents out of any point of the double tangent lie on a conic through the cusps.

For $y_3 = 0$, so $y_1 : y_2 = b_2 : -b_1$ (the point H) we find as it ought to be

$$(b_1x_1 - b_2x_2)\,x_3 = 0.$$

The conics ξ_2 form evidently a pencil of which two basepoints lie on h, the remaining two in O_1 and O_2.

7. The curve of HESSE of C_4 has as equation.

$$6x_1^3x_2^3 + 18(b_1x_1 + b_2x_2)\,x_1^2x_2^2x_3 + (18c + 32)\,x_1^2x_2^2x_3^2 +$$
$$+ 60(b_1x_1 + b_2x_2)\,x_1x_2x_3^3 + (36b_1b_2 + 24c - 8)\,x_1x_2x_3^4 +$$
$$+ 9(b_1x_1 + b_2x_2)^2\,x_3^4 + 18(b_1x_1 + b_2x_2)\,x_3^5 + (18b_1b_2 + c)\,x_3^6 = 0.$$

By combination with the equation of C_4 we find that the points of intersection of the two curves not lying in the cusps are situated on the curve

$$12(b_1x_1 + b_2x_2)x_1x_2 + (18b_1b_2 - 18c - 30)x_1x_2x_3 - 27(b_1x_1 + b_2x_2)^2x_3 -$$
$$-(54c + 22)(b_1x_1 + b_2x_2)x_3^2 + (18b_1b_2 - 19c - 18c^2)x_3^3 = 0.$$

So the eight points of inflexion of the C_4 are situated on a cubic curve passing through the cusps and the point H.

The polarcurve of the point $O_3 = k_1\,k_2$ consists of $x_3 = 0$ and the conic

$$2\,x_1x_2 + 3\,b_1x_1x_3 + 3\,b_2x_2x_3 + 2\,cx_3^2 = 0,$$

passing through the cusps and through the points of contact of the four tangents which meet in the point of concurrence of the cuspidal tangents.

It is easy to see that O_3 and H are the only points for which the polarcurve degenerates.

Chemistry. — *"On the system hydrogen bromide and bromine."* By Dr. E. H. BÜCHNER and Dr. B. J. KARSTEN. (Communicated by Prof. A. F. HOLLEMAN).

The research, a report of which is given here, was undertaken in connection with a remark from Prof. HOLLEMAN, that the existence of compounds of the type HBr_n has been assumed several times in order to explain the mechanism of reactions in organic chemistry. In order to test the validity of this assumption it was thought desirable to ascertain, in the first place, whether pure bromine and