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TABLE VIII (continued).

x	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
5.0	0.9596	—0.0812	—0.0562	0.0211	0.0140
5.1	0.9628	—0.0793	—0.0555	0.0184	0.0151
5.2	0.9658	—0.0746	—0.0547	0.0159	0.0162
5.3	0.9686	—0.0701	—0.0538	0.0135	0.0169
5.4	0.9711	—0.0659	—0.0527	0.0112	0.0175
5.5	0.9734	—0.0618	—0.0515	0.0090	0.0179
5.6	0.9756	—0.0580	—0.0503	0.0070	0.0182
5.7	0.9776	—0.0544	—0.0489	0.0050	**0.0183**
5.8	0.9794	—0.0509	—0.0475	0.0032	0.0182
5.9	0.9811	—0.0477	—0.0461	0.0016	0.0181
6.0	0.9827	—0.0446	—0.0446	0.0000	0.0178
6.5	0.9887	—0.0318	—0.0371	—0.0060	0.0154
7.0	0.9927	—0.0223	—0.0298	—0.0093	0.0119
7.5	0.9953	—0.0136	—0.0233	—0.0107	0.0083
8.0	0.9970	—0.0107	—0.0179	**—0.0107**	0.0050
8.5	0.9981	—0.0074	—0.0135	—0.0100	0.0024
9.0	0.9988	—0.0050	—0.0100	—0.0088	0.0005
9.5	0.9992	—0.0034	—0.0073	—0.0074	—0.0008
10.0	0.9995	—0.0023	—0.0053	—0.0061	—0.0015
10.5	0.9997	—0.0015	—0.0038	—0.0048	—0.0019
11.0	0.9998	—0.0010	—0.0027	—0.0038	**—0.0020**
11.5	0.9999	—0.0007	—0.0019	—0.0029	—0.0019
12.0	0.9999	—0.0004	—0.0013	—0.0022	—0.0017
12.5	1.0000	—0.0003	—0.0009	—0.0017	—0.0015
13.0	—	—0.0002	—0.0006	—0.0012	—0.0012
13.5	—	—0.0001	—0.0004	—0.0009	—0.0010
14.0	—	—0.0001	—0.0003	—0.0007	—0.0008
14.5	—	—0.0001	—0.0002	—0.0005	—0.0007
15.0	—	0.0000	—0.0001	—0.0003	—0.0005
15.5	—	—	—0.0001	—0.0002	—0.0004
16.0	—	—	—0.0001	—0.0002	—0.0003
16.5	—	—	0.0000	—0.0001	—0.0002
17.0	—	—	—	—0.0001	—0.0002
17.5	—	—	—	—0.0001	—0.0001
18.0	—	—	—	0.0000	—0.0001
18.5	—	—	—	—	—0.0001
19.0	—	—	—	—	0.0000

Mathematics. — *"On curves of order four with two flecnodal points or with two biflecnodal points."* By Prof. JAN DE VRIES.

1. The points of a binodal curve of order four, C_4, are projected out of the two double points O_1 and O_2 by two pencils in correspondence (2 , 2).

So such a C_4 is determined by the relation

$$a_{22}\lambda^2\mu^2+a_{21}\lambda^2\mu+a_{20}\lambda^2+a_{12}\lambda\mu^2+a_{11}\lambda\mu+a_{10}\lambda+a_{02}\mu^2+a_{01}\mu+a_{00}=0,$$

where

$$\lambda = x_1 : x_3 \quad \text{and} \quad \mu = x_2 : x_3 .$$

According to a well-known property the eight singular rays (λ)

are in four ways projective to the eight singular rays (μ); consequently through O_1 and O_2 pass four conics bearing each four points of intersection of two tangents out of O_1 and O_2 and at the same time four points of intersection of rays out of O_1 and O_2 to the points of contact of those tangents (double-rays of the (2, 2)).

If O_1O_2 is a branchray for both pencils, one of the four conics degenerates, in which case C_4 has cusps in O_1 and O_2 (see my paper "*On bicuspidal curves of order four*", Proceedings of the meeting of Dec. 24th 1908, Vol. IX, p. 499).

We suppose that O_1O_2 is conjugate as double-ray to the branch-rays O_1O_3 and O_2O_3. The equation of correspondence must then furnish for $\lambda = 0$ and for $\mu = 0$ the equations $\mu^2 = \infty$ and $\lambda^2 = \infty$; hence $a_{20} = 0$, $a_{02} = 0$, $a_{10} = 0$, $a_{01} = 0$.

The equation of C_4 can now be written in the form

$$x_1^2 x_2^2 + 2x_1x_2x_3(b_1x_1 + b_2x_2 + b_3x_3) + x_3^4 = 0.$$

In each of the two double points one of the branches has an inflectional point; the corresponding tangents are $x_1 = 0$ and $x_2 = 0$.

Out of each of the two *flecnodal points* three more tangents can be drawn to C_4. They are represented by

$$b_1^2x_1^3 + 2b_1b_3x_1^2x_3 + (b_3^2-1)\,x_1x_3^2 - 2b_2x_3^3 = 0,$$
$$b_2^2x_2^3 + 2b_2b_3x_2^2x_3 + (b_3^2-1)\,x_2x_3^2 - 2b_1x_3^3 = 0.$$

By eliminating x_3^3 we find

$$(b_1^3x_1^3 - b_2^3x_2^3) + 2b_3x_3\,(b_1^2x_1^2 - b_2^2x_2^2) + (b_3^2-1)\,x_3^2\,(b_1x_1 - b_2x_2) = 0.$$

So on the right line $b_1x_1 = b_2x_2$ lie three points of intersection of the tangents out of O_1 with the tangents out of O_2. We shall indicate it by h.

It is evident that these three points and the point O_3 are the branchpoints for the two collocal series of points in correspondence (2,2), determined by the pencils (O_1) and (O_2) on the line h. So according to a well-known property this (2,2) is involutory.

Indeed, we find out of

$$\lambda^2\,\mu^2 + 2\,b_1\,\lambda^2\,\mu + 2\,b_2\,\lambda\,\mu^2 + 2\,b_3\,\lambda\,\mu + 1 = 0$$

and

$$b_1\,\lambda = b_2\,\mu',$$

that the (2,2) is indicated on h by the symmetric relation

$$b_2^2\,\mu^2\,\mu'^2 + 2\,b_1\,b_2^2\,(\mu^2\,\mu' + \mu\,\mu'^2) + 2\,b_1\,b_2\,b_3\,\mu\,\mu' + b_1^2 = 0$$

between the rays projecting it out of O_2.

2. If Q, Q' is a pair of the involutory (2,2) on h, then the points $P_1 \equiv (O_1Q, O_2Q')$ and $P_2 \equiv (O_1Q', O_2Q)$ lie on C_4. The line P_1P_2

intersects $O_1 O_2$ in a point H, separated harmonically by the line h from O_1 and O_2.

So the pairs of points P_1, P_2 form on C_4 a *fundamental involution* F_2, of which each ray through H contains two pairs.

The coincidences of F_2 are the points of contact of the tangents out of $H (y_1 = b_2, y_2 = - b_1, y_3 = 0)$. The polar curve of H has as equation

$$(b_1 x_1 - b_2 x_2)(x_1 x_2 + x_3 b_x) = 0,$$

so it consists of the line h and the conic

$$x_1 x_2 + x_3 b_x = 0.$$

The points of intersection of this conic with C_4,

$$x_1^2 x_2^2 + 2 x_1 x_2 x_3 b_x + x_3^4 = 0,$$

lie on $x_3^2 = 0$ and on $x_3^2 = b_x^2$.

By combining

$$b_x = \pm x_3$$

with the equation of C_4 we find $(x_1 x_2 \pm x_3^2)^2 = 0$. So H is the point of intersection of two double tangents.

The points of contact of these double tangents forming two pairs of F_2 and being generated by the conics $x_1 x_2 \pm x_3^2 = 0$, the supposition is at hand that F_2 can also be determined by means of the pencil of conics

$$x_1 x_2 = \varrho x_3^2.$$

Indeed, the movable points of intersection of these conics with C_4 lie on the rays

$$(1 + \varrho^2) x_3 + 2 \varrho b_x = 0,$$

passing through H, whilst the line h,

$$b_1 x_1 = b_2 x_2,$$

is the polar of H with respect to each conic

$$x_1 x_2 = \varrho x_3^2.$$

Resuming we can say:

Of a C_4 with two flecnodal points O_1 and O_2 two double tangents meet on the connecting line $O_1 O_2$ of the double points. The points of contact of the four tangents which it is possible still to draw out of their point of intersection to C_4 lie on a right line, which contains moreover three points of intersection of the tangents r_1, s_1, t_1 out of O_1 with the tangents r_2, s_2, t_2 out of O_2 and the point of intersection of the inflectional tangents f_1 and f_2 in O_1 and O_2.

3. From $(f_1 r_1 s_1 t_1) = (f_2 r_2 s_2 t_2)$ follows

$$(f_1 r_1 s_1 t_1) = (r_2 f_2 t_2 s_2) = (s_2 t_2 f_2 r_2) = (t_2 s_2 r_2 f_2).$$

By this three conics $\varrho_2, \sigma_2, \tau_2$ through O_1 and O_2 are determined containing in succession the quadruplets of points

$$f_1r_2 \quad , \quad r_1f_2 \quad , \quad s_1t_2 \quad , \quad t_1s_2 \, ;$$
$$f_1s_2 \quad , \quad r_1t_2 \quad , \quad s_1f_2 \quad , \quad t_1r_2 \, ;$$
$$f_1t_2 \quad , \quad r_1s_2 \quad , \quad s_1r_2 \quad , \quad t_1f_2 \, .$$ [1])

On these too the pencils (O_1) and (O_2) arranged in (2, 2) determine involutory (2,2), which then again are connected with *fundamental involutions* on C_4. The pairs of such an involution lie on rays through the pole R, S, T of O_1O_2 with respect to the corresponding conic $\varrho_2, \sigma_2, \tau_2$. This pole is the point of intersection of two double tangents; this follows amongst others from the fact, that the point of contact of each tangent of the C_4 drawn from R must lie on the conic ϱ_2 and must be a coincidence of the involutory (2,2); the number of these tangents amounts thus to four, so that the remaining tangents out of R must coincide two by two in two double tangents.

For further particulars about the properties which can be deduced from these observations I refer to my paper mentioned above and to the paper named in it published in "N. Archief voor Wiskunde, XIV."

4. We shall now suppose that O_1 and O_2 are *biflecnodal points*. Let us choose the point O_3 in such a way, that the tangents in O_1 and in O_2 are separated harmonically by O_1O_2, O_1O_3 resp. by O_2O_1, O_2O_3, then the equation of C_4 has the form

$$x_1^2x_2^2 - a_1^2x_2^2x_3^2 - a_2^2x_1^2x_3^2 + b_0x_1x_2x_3^2 + b_1x_1x_3^3 + b_2x_2x_3^3 + c^2x_3^4 = 0.$$

If O_1 and O_2 are to become biflecnodal points, then we shall every time have to find when substituting $x_1 = \pm a_1x_3$ and $x_2 = \pm a_2x_3$ that $x_3^4 = 0$. For this is necessary $b_2 \pm a_1b_0 = 0$ and $b_1 \pm a_2b_0 = 0$, thus $b_0 = 0$, $b_1 = 0$ and $b_2 = 0$ [2]).

So we have to deal with the equation

$$x_1^2x_2^2 - a_1^2x_2^2x_3^2 - a_2^2x_1^2x_3^2 + c^2x_3^4 = 0.$$

If we write for this

$$(x_1^2 - a_1^2x_3^2)(x_2^2 - a_2^2x_3^2) + (c^2 - a_1^2a_2^2)\, x_3^4 = 0,$$

and if we put moreover

$$c^2 - a_1^2a_2^2 = f^2,$$

it is evident that C_4 can be generated by the projective involutions of rays

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[1]) The six points r_1s_2, s_1r_2, s_1t_2, t_1s_2, t_1r_2, r_1t_2 lie on a conic; for, through r_1r_2, s_1s_2, t_1t_2 passes the line h.

[2]) We find moreover that C_4 cannot have at the same time a flecnodal point and a biflecnodal point.

$$\left. \begin{aligned} x_1^2 - a_1^2 x_3^2 &= \lambda f x_3^2 \\ f x_3^2 &= -\lambda (x_2^2 - a_2^2 x_3^2). \end{aligned} \right\}$$

In this C_4 thus ∞^1 *quadrangles are described having all O_1 and O_2 as diagonal points.*

The vertices of these quadrangles evidently form a *fundamental involution F_4.*

Out of

$$\left. \begin{aligned} x_1^2 &= (a_1^2 + \lambda f)\, x_3^2 , \\ \lambda x_2^2 &= (\lambda a_2^2 - f)\, x_3^2 \end{aligned} \right\}$$

we find for the diagonals of the quadrangle (λ) the equation

$$(\lambda a_2^2 - f)\, x_1^2 = (\lambda a_1^2 + \lambda^2 f)\, x_2^2 .$$

So all quadrangles have in O_3 their *third diagonal point.*

At the same time it is evident from this that we can build up the above mentioned F_4 out of pairs of the fundamental F_2 of which each ray through O_3 contains two pairs.

If the two pairs coincide then the ray which bears them is a double tangent.

The pairs on the ray $x_1 = \varrho x_2$ we find out of

$$f\lambda^2 + (a_1^2 - a_2^2 \varrho^2)\, \lambda + f\varrho^2 = 0$$

Thus for a double tangent we have

$$(a_1^2 - a_2^2 \varrho^2)^2 = 4 f^2 \varrho^2 ,$$

or

$$a_2^2 \varrho^2 \pm 2 f \varrho - a_1^2 = 0.$$

So O_3 is the point of intersection of four double tangents corresponding to

$$a_2^2 x_1^2 \pm 2 f x_1 x_2 - a_1^2 x_2^2 = 0,$$

or, what comes to the same, to

$$a_2^2 x_1^2 \pm 2 c x_1 x_2 + a_1^2 x_2^2 = 0.$$

The eight points of contact lie on a conic.

For, the polar curve of O_3 degenerates into $x_3 = 0$ and the conic

$$a_2^2 x_1^2 + a_1^2 x_2^2 - 2 c^2 x_3^2 = 0.$$

5. We shall show now that the remaining four double tangents are connected with two *fundamental involutions of pairs* which can be generated by conics.

The curve C_4 can be generated by the projective pencils

$$\left. \begin{aligned} (x_1 - a_1 x_3)(x_2 - a_2 x_3) &= \varrho f x_3^2 , \\ \varrho (x_1 + a_1 x_3)(x_2 + a_2 x_3) &= -f x_3^2 . \end{aligned} \right\}$$

Evidently the two variable points of intersection of conjugate conics lie on the line

$$2\varrho\,(a_1x_2 + a_2x_1) + (\varrho^2 + 1)fx_3 = 0,$$

passing through the point H_1 having as coordinates $(a_1, -a_2, 0)$.

Each line

$$a_1x_2 + a_2x_1 + \sigma fx_3 = 0$$

bears two pairs of the fundamental involution which can be generated by each of the two pencils of conics; for we have $\varrho^2 - 2\sigma\varrho + 1 = 0$.

For $\varrho = \pm 1$ these pairs coincide and we find the *double tangents*

$$a_1x_2 + a_2x_1 \pm fx_3 = 0.$$

In a similar way the pencils

$$\left.\begin{aligned}(x_1 - a_1x_3)\,(x_2 + a_2x_3) &= \varrho fx_3^2,\\ \varrho\,(x_1 + a_1x_3)\,(x_2 - a_2x_3) &= -fx_3^2\end{aligned}\right\}$$

determine a fundamental involution which is also generated by the rays out of the point $H_2\,(a_1, a_2, 0)$, through which at the same time the double tangents

$$a_1x_2 - a_2x_1 \pm fx_3 = 0$$

pass.

The *four double tangents* form a quadrilateral having $O_1O_2O_3$ as diagonal triangle.

6. The polar line of $(a_1, \pm a_2, 0)$ with respect to the conic

$$(x_1 - a_1x_3)\,(x_2 \pm a_2x_3) = \varrho fx_3^2$$

is represented by

$$a_1x_2 \pm a_2x_1 = 0.$$

From this ensues that the pencils (H_1) and (H_2) determine two involutory (2,2) on these two lines h_1 and h_2. Their branchpoints are generated by the nodal tangents and the tangents which can still be drawn out of O_1 and O_2.

If we write the equation of C_4 in the form

$$(x_1^2 - a_1^2x_3^2)\,x_2^2 - (a_2^2x_1^2 - c^2x_3^2)\,x_3^2 = 0,$$

it is evident that the lines

$$a_2^2x_1^2 = c^2x_3^2$$

touch it on $x_2 = 0$.

In an analogous way the lines

$$a_1^2x_2^2 = c^2x_3^2$$

have their points of contact on $x_1 = 0$.

And now we see directly that these two pairs of rays intersect each other on the lines h_1 and h_2,

$$a_1x_2 \pm a_2x_1 = 0,$$

which bear at the same time the points of intersection of the nodal tangents

$$x_1^2 = a_1^2 x_2^2 \text{ and } x_2^2 = a_2^2 x_3^2.$$

The remaining points of intersection of the two fourrays lie on the conic

$$a_1^2 x_2^2 + a_2^2 x_1^2 - (a_1^2 a_2^2 + c^2)\, x_3^2 = 0.$$

This is immediately evident, if we eliminate out of the equations

$$(a_1^2 x_2^2 - c^2 x_3^2)\,(x_2^2 - a_2^2 x_3^2) = 0,$$
$$(a_2^2 x_1^2 - c^2 x_3^2)\,(x_1^2 - a_1^2 x_3^2) = 0$$

the quantity x_3^4.

The coincidences on h_1 and h_2 here also originate from the tangents out of H_1 and H_2. Indeed we find for the polar curves of H_1 and H_2

$$a_1(x_1 x_2^2 - a_2^2 x_1 x_3^2) \pm a_2(x_1^2 x_2 - a_1^2 x_2 x_3^2) = 0,$$

or

$$(a_1 x_1 \pm a_2 x_2)\,(x_1 x_2 \mp a_1 a_2 x_3^2) = 0.$$

From this is evident at the same time that the conics

$$x_1 x_2 \mp a_1 a_2 x_3^2 = 0$$

generate the points of contact of the double tangents meeting in H_1 and H_2.

By combining the equation

$$x_1^2 x_2^2 - a_1^2 a_2^2 x_3^4 = 0$$

with the equation of C_4 we find that the eight points of contact of the four double tangents are situated on the conic

$$a_2^2 x_1^2 + a_1^2 x_2^2 = (a_1^2 a_2^2 + c^2)\, x_3^2$$

7. The curve of HESSE is represented by

$$(a_1^2 x_2^2 + a_2^2 x_1^2)\, x_1^2 x_2^2 + \{(8 a_1^2 a_2^2 - 6c^2)\, x_1^2 x_2^2 - (a_1^2 x_2^2 + a_2^2 x_1^2)^2\}\, x_3^2 +$$
$$+ (a_1^2 a_2^2 - 2c^2)\,(a_1^2 x_2^2 + a_2^2 x_1^2)\, x_3^4 + 2 a_1^2 a_2^2\, c^2 x_3^6 = 0.$$

If we eliminate $x_1^2 x_2^2$ out of this equation and the equation of C_4,

$$x_1^2 x_2^2 - (a_1^2 x_2^2 + a_2^2 x_1^2)\, x_3^2 + c^2 x_3^4 = 0,$$

it is evident that the points which the two curves have in common besides O_1 and O_2 are situated on the conic

$$3\,(a_2^2 x_1^2 + a_1^2 x_2^2) = 2c^2 x_3^2.$$

The eight points of inflexion of a C_4 with two biflecnodal points are points of intersection with a conic.

They lie two by two on four right lines through the point of intersection O_3 of the four double tangents of the first group.

The polarcurve η_3 of the point (y) is represented by

$$(y_1 x_2 + y_2 x_1) x_1 x_2 - y_3 (a_1^2 x_2^2 + a_2^2 x_1^2) x_3 - (a_1^2 y_2 x_2 + a_2^2 y_1 x_1) x_3^2 + 2c^2 y_3 x_3^3 = 0.$$

As it is touched in O_1 and O_2 by the lines

$$y_2x_2 - a^2{}_2y_3x_3 = 0 \text{ and } y_1x_1 - a_1{}^2y_3x_3 = 0,$$

we find that

$$y_1y_2x_1x_2 - a_1{}^2y_2y_3x_2x_3 - a_2{}^2y_1y_3x_1x_3 + c^2y_3{}^2x_3{}^2 = 0$$

represents a conic η_2 touching the polar curve in O_1 and O_2.

If (y) lies on C_4, then

$$y_1{}^2y_2{}^2 - a_1{}^2y_2{}^2y_3{}^2 - a_2{}^2y_1{}^2y_3{}^2 + c^2y_3{}^4 = 0,$$

i. e. (y) also belongs to η_2. The tangent (y) to η_2 has as equation

$$y_1y_2(y_1x_2+y_2x_1) - (a_1{}^2y_2{}^2+a_2{}^2y_1{}^2)y_3x_3 - y_3{}^2(a_1{}^2y_2x_2+a_2{}^2y_1x_1)+2c^2y_3{}^3x_3 = 0.$$

As when (x) and (y) are exchanged it determines the polar curve η_3 it represents at the same time the tangent in (y) to C_4.

In each of its points C_4 is touched by a conic which touches the polar curve of that point in the biflecnodal points.

The curves C_4 and η_2 have two more points in common. If l is their connecting line, then the pencil determined by C_4 and $\eta_3 + l$ contains a curve composed of η_2 and a second conic. From this ensues: *The points of contact of the six tangents out of a point of* C_4 *can be connected by a conic.*

8. The projective involutions of rays (O_1) and (O_2) have as double rays

$$\left.\begin{array}{ll}\lambda = \infty, & x_3{}^2 = 0,\\ \lambda = -a_1{}^2 : f, & x_1{}^2 = 0.\end{array}\right\} \text{ and } \left\{\begin{array}{ll}\lambda = 0, & x_3{}^2 = 0,\\ \lambda = f : a^2{}_2, & x_2{}^2 = 0.\end{array}\right.$$

When the double rays O_2O_3 and O_1O_3 are conjugated to each other, their point of intersection becomes a third double point of C_4. This takes place when we have

$$\frac{f}{a_2{}^2} + \frac{a_1{}^2}{f} = 0, \text{ or } c^2 = 0.$$

The C_4 is then represented by

$$x_1{}^2x_2{}^2 - a_1{}^2x_2{}^2x_3{}^2 - a_2{}^2x_1{}^2x_3{}^2 = 0.$$

So it has *three biflecnodal points.* As is evident from the above we can describe in this C_4 ∞ quadrangles having the three double points as diagonal points.

The double tangents of the first group are now replaced by the tangents in O_3 (§ 4). In each of the biflecnodal points the tangents are harmonically separated by the lines to the remaining two double points.

The C_4 with three biflecnodal points have been extensively treated by LAGUERRE (*Nouv. Ann.* 2e série XVII, 1878) and by SCHOUTE (*Archiv der Math. und Phys.* 2e Reihe, II, III, IV, VI, 1885–87).