**Mathematics.** — *"On curves which can be generated by projective involutions of rays."* By Prof. JAN DE VRIES.

1. By the symbol

$$(a_1x_1 + a_2x_2)^{(n)}$$

we shall indicate a homogeneous form of order $n$.

By the projective involutions of rays

$$(a_2x_2 + a_3x_3)^{(n)} + \lambda\,(\alpha_2x_2 + \alpha_3x_3)^{(n)} = 0,$$
$$(b_1x_1 + b_3x_3)^{(n)} + \lambda\,(\beta_1x_1 + \beta_3x_3)^{(n)} = 0$$

a curve $C_{2n}$ is generated in which $\infty^1$ $2n$-sides are described possessing in $O_1$ and $O_2$ $n$-fold vertices. For brevity I call such a $2n$-side *bisingular.*

$O_1$ and $O_2$ are $n$-fold points of the curve. The tangents in $O_1$ form a group of the first involution which is conjugated to the group of the second containing the ray $O_2O_1$. These two groups determine a singular $2n$-side, where $O_1$ replaces $\frac{1}{2}\,n\,(n+1)$, and $O_2$ replaces $\frac{1}{2}\,n\,(n-1)$ vertices.

*If we can describe in a $C_{2n}$ with two $n$-fold points one bisingular $2n$-side it bears an infinite number of those figures.*

For, if the indicated $2n$-side is represented by the two groups of rays

$$(a_2x_2 + a_3x_3)^{(n)} = 0 \quad , \quad (b_1x_1 + b_3x_3)^{(n)} = 0\,,$$

and if $x_3 = mx_2$ is one of the rays of the first group, then the substitution must furnish $x_2^{\,n}\,(b_1x_1 + b_3mx_2)^{(n)} = 0$, $x_3 = mx_2$, $O_1$ being an $n$-fold point. Hence the equation of $C_{2n}$ must have the form

$$(a_2x_2 + a_3x_3)^{(n)}\,(\beta_1x_1 + \beta_3x_3)^{(n)} = (b_1x_1 + b_3x_3)^{(n)}\,(\alpha_2x_2 + \alpha_3x_3)^{(n)}\,. \quad (1)$$

But then the equation can be formed by elimination out of

$$\left.\begin{array}{l}(a_2x_2 + a_3x_3)^{(n)} + \lambda\,(\alpha_2x_2 + \alpha_3x_3)^{(n)} = 0,\\ (b_1x_1 + b_3x_3)^{(n)} + \lambda\,(\beta_1x_1 + \beta_3x_3)^{(n)} = 0,\end{array}\right\}\quad .\;.\;.\;.\quad (2)$$

and the curve contains the $\infty^1$ bisingular $2n$-sides which can be indicated by these two equations, $\lambda$ varying.

2. We shall now investigate under which condition two projective involutions of rays will generate a curve $C_{2n}$ with three $n$-fold points $O_k$, so that $n^2$ points of intersection of two conjugate groups of rays are vertices of three different bisingular $2n$-sides having each two of the points $O_k$ as $n$-fold vertices.

In that case we must be able to bring through the points of intersection of

$$(a_2x_2 + a_3x_3)^{(n)} = 0 \quad \text{and} \quad (b_1x_1 + b_3x_3)^{(n)} = 0$$

a group of rays

$$(c_1x_1 + c_2x_2)^{(n)} = 0$$

It is now at once evident that this is only possible when the first two equations have the following form

$$a_2{}^nx_2{}^n - a_3{}^nx_3{}^n = 0, \quad a_1{}^nx_1{}^n - a_3{}^nx_3{}^n = 0,$$

so that we have

$$a_1{}^nx_1{}^n - a_2{}^nx_2{}^n = 0.$$

Out of

$$a_2{}^nx_2{}^n - a_3{}^nx_3{}^n + \lambda\,(\alpha_2{}^nx_2{}^n - \alpha_3{}^nx_3{}^n) = 0$$

and

$$a_1{}^nx_1{}^n - a_3{}^nx_3{}^n + \lambda\,(\alpha_1{}^nx_1{}^n - \alpha_3{}^nx_3{}^n) = 0$$

follows

$$(a_2{}^nx_2{}^n - a_3{}^nx_3{}^n)\ (\alpha_1{}^nx_1{}^n - \alpha_3{}^nx_3{}^n) - (a_1{}^nx_1{}^n - a_3{}^nx_3{}^n)\ (\alpha_2{}^nx_2{}^n - \alpha_3{}^nx_3{}^n) = 0$$

or in transparent notation

$$(a\alpha)_3x_1{}^nx_2{}^n + (a\alpha)_1x_2{}^nx_3{}^n + (a\alpha)_2x_3{}^nx_1{}^n = 0. \quad . \quad . \quad . \quad (3)$$

The tangents in $O_3$ are represented by

$$(a\alpha)_1x_2{}^n + (a\alpha)_2x_1{}^n = 0.$$

If $x_2 = kx_1$ is the equation of one of these tangents, then the substitution in the equation of the $C_{2n}$ evidently furnishes $x_1{}^{2n} = 0$. In each of the $n$-fold points each tangent has thus $(n+1)$ points in common with the corresponding branch.

For each value of $\lambda$ we find a figure consisting of $3n$ lines (of which however only 3 or 6 are real, according to $n$ being even or odd) and $(n^2 + 3)$ points (of which 4 or 7 are real). [1])

1) We have in particular for $n = 3$ a configuration $(12_3, 9_4)$. From this ensues, by the way, that of the configuration $(9_4, 12_3)$ corresponding dually to it only 3 points and 4 lines can be real. From the above it is evident that the 12 lines of the $(9_4, 12_3)$ can be represented by

$$\left.\begin{array}{l} \xi_1 = 0,\ \xi_2 = 0 \\ \xi_2 = 0,\ \xi_3 = 0 \\ \xi_3 = 0,\ \xi_1 = 0 \end{array}\right\} \text{ and } \xi_1 = \varepsilon\xi_2 = \varepsilon^2\xi_3, \text{ where } \varepsilon^3 = 1 \text{ is.}$$

The three lines $\xi_3 = \xi_2 = \xi_1$, $\xi_3 = \varepsilon\xi_2 = \varepsilon^2\xi_1$, $\xi_2 = \varepsilon^2\xi_1 = \varepsilon\xi_3$ contain together the 9 points. They are also indicated by

$$x_1 + x_2 + x_3 = 0,\ x_1 + \varepsilon x_2 + \varepsilon^2 x_3 = 0,\ x_1 + \varepsilon^2 x_2 + \varepsilon x_3 = 0.$$

The 9 points lying also on $x_1x_2x_3 = 0$, they are the base-points of the pencil

$$(x_1 + x_2 + x_3)\ (x_1 + \varepsilon x_2 + \varepsilon^2 x_3)\ (x_1 + \varepsilon^2 x_2 + \varepsilon x_3) + m\,x_1x_2x_3 = 0.$$

And so here we have found back the canonical equation of $C_3$.

3. The projective involutions of rays

$$\left.\begin{array}{l}(a_2x_2 + a_3x_3)^{(n)} + \lambda x_3^k (\alpha_2x_2 + \alpha_3x_3)^{(n-k)} = 0, \\ (b_1x_1 + b_3x_3)^{(n)} + \lambda x_3^k (\beta_1x_1 + \beta_3x_3)^{(n-k)} = 0,\end{array}\right\} \quad . \quad . \quad . \quad (4)$$

generate evidently a $C_{2n-k}$, which has $O_1$ and $O_2$ as $(n-k)$-fold points and as equation

$$(a_2x_2 + a_3x_3)^{(n)} (\beta_1x_1 + \beta_3x_3)^{(n-k)} = (b_1x_1 + b_3x_3)^{(n)} (\alpha_2x_2 + \alpha_3x_3)^{(n-k)} \ . \ (5)$$

The two multiple points are for $k > 1$ of a particular kind. For the tangents in $O_1$ are represented by $(\alpha_2x_2 + \alpha_3x_3)^{(n-k)} = 0$, and each of them has as is evident from substitution $(k+1)$ points in common with the corresponding branch of the curve.

For $x_3 = 0$ we find

$$x_2^{n-k} x_2^{n-k} (a_0\beta_0 x_2^k - b_0\alpha_0 x_1^k) = 0.$$

Therefore the curve is intersected by $O_1O_2$ in a group of the involution $I_k$ which has $O_1$ and $O_2$ as $k$-fold points.

If we can describe in a $C_{2n-k}$ with two $(n-k)$-fold points *a bisingular* $2n$-*side* having those multiple points as $n$-fold points it has an equation of form (5). But then it can be generated by two involutions of form (4) and it bears therefore $\infty$ *bisingular* $2n$-*sides*.

4. For $k = n$ we find a $C_n$ which will in general not possess any singular points. Yet it is in general not possible to generate a $C_n$ by two involutions of rays of order $n$. For, the centres $O_1$ and $O_2$ of the involutions must be $n$-fold points of an involution $I_n$, of which the points of intersection of $C_n$ with $O_1O_2$ form a group. But then the polar curve of $O_1$ would have to have $(n-1)$ points in $O_2$ in common with the right line $O_1O_2$, and this is not possible for a general $C_n$.

But each *cubic curve* can be generated by two projective cubic involutions of rays. Their centres $O_1$ and $O_2$ are conjugate points of the curve of Hesse, for the two double rays which $O_1$ possesses (besides the threefold ray $O_1O_2$), bearing each of them the points of contact of three tangents out of $O_2$, form the polar conic of $O_2$, whilst the rays which complete the two double rays to groups of the involution form the satellite conic of $O_2$.

Let us now take inversely $O_1$ and $O_2$ as two conjugate points of the curve of Hesse. We regard $O_2$ as centre of a cubic involution which has $O_2O_1$ as threefold element, whilst a second group is formed by three tangents the points of contact of which lie in a line $r$, so that their points of intersection with $C_3$ are situated on a line $s$. The line $r$ counted double and the line $s$ we unite to a group of a cubic involution $(O_1)$ having $O_1O_2$ as threefold ray. We now make the two involutions projective in such a way that the threefold rays correspond, that the group $(\bar{r}rs)$ is conjugated to the group of the three tangents and that finally the groups are assigned to each other which are determined by the rays to an arbitrary point of $C_3$. The two involutions then generate a $C_3$ having with the given $C_3$ ten points in common, thus coinciding with it.

*In each general cubic curve we can thus describe* $\infty^2$ *bisingular hexagons.*

*Their threefold points lie on the curve of* HESSE.

5. If the ray $O_1O_2$ counted double belongs to corresponding groups of the cubic involutions $(O_1)$ and $(O_2)$, these involutions generate a $C_4$ which has $O_1$ and $O_2$ as points of inflection the tangents of which meet each other on the curve. For, out of

$$a_0x_2^3 + 3a_1x_2^2x_3 + 3a_2x_2x_3^2 + a_3x_3^3 + \lambda x_2x_3^2 = 0,$$
$$b_0x_1^3 + 3b_1x_1^2x_3 + 3b_2x_1x_3^2 + b_3x_3^3 + \lambda x_1x_3^2 = 0$$

we find

$$(a_0x_2^3+3a_1x_2^2x_3+3a_2x_2x_3^2+a_3x_3^3)x_1 = (b_0x_1^3+3b_1x_1^2x_3+3b_2x_1x_3^2+b_3x_3^3)x_2,$$

and this is satisfied by

$$x_1 = 0,\ x_2x_3^3 = 0 \text{ and } x_2 = 0,\ x_1x_3^3 = 0.$$

According to the rule found in § 3 $O_1O_2$ is harmonically divided by $C_4$.

Inversely, *when two stationary tangents of a* $C_4$ *intersect each other on the curve whilst their points of contact are harmonically separated by* $C_4$, *then those points are threefold vertices of* $\infty^1$ *bisingular hexagons described in* $C_4$.

For, in that case the equation of $C_4$ has the form

$$(c_1x_1^2 + c_2x_2^2)x_1x_2 + (f_1x_1 + f_2x_2 + f_3x_3)\,x_1x_2x_3 + (g_1x_1 + g_2x_2)x_3^3 = 0.$$

If we replace it by

$$\{c_1x_1^3 + f_1x_1^2x_3 + (\tfrac{1}{2}f_3 + \varrho)x_1x_3^2 + g_2x_3^3\}x_2 +$$
$$\{c_2x_2^3 + f_2x_2^2x_3 + (\tfrac{1}{2}f_3 - \varrho)x_2x_3^2 + g_1x_3^3\}\,x_1 = 0,$$

it is evident, that the curve can be generated by the pencils

$$\left.\begin{array}{l} c_1x_1^3 + f_1x_1^2x_3 + (\tfrac{1}{2}f_3 + \varrho)\,x_1x_3^2 + g_2x_3^3 + \lambda x_1x_3^2 = 0, \\ c_2x_2^3 + f_2x_2^2x_3 + (\tfrac{1}{2}f_3 - \varrho)\,x_2x_3^2 + g_1x_3^3 - \lambda x_2x_3^2 = 0. \end{array}\right\}$$

Here we can still replace $(\lambda + \varrho)$ by $\mu$.

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