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Mathematics. — *"The plane curve of order 4 with 2 or 3 cusps and 0 or 1 nodes as a projection of the twisted curve of order 4 and of the 1st species."* By Prof. H. DE VRIES.

(Communicated in the meeting of January 30 1909.)

1. If two quadratic cones are situated arbitrarily with respect to each other, they intersect each other in a twisted curve r^4 of order 4 and of the 1st species. If we suppose the plane τ to be brought through a point O of the nodal curve of the developable belonging to r^4 in which plane lie the two tangents of r^4 passing through O, then this plane must intersect the two cones according to conics k^2_1, k^2_2, touching each other in the points of contact O_1, O_2 of the two indicated tangents with r^4. We shall now suppose the first cone to be determined by the base-curve k^2_1 and the vertex R, the second by k^2_2 and the vertex S. The plane τ is a double tangential plane of r^4, so it must be a tangential plane of *one* of the four quadratic cones passing through r^4; i. e. in τ, and on the line O_1O_2, lies the vertex H of a third double projecting cone of r^4; and finally the vertex T of the fourth cone must then lie in the common polar plane of H with respect to the cones $[R]$ and $[S]$, and this plane must pass through O, because the double curve of the developable of r^4 consists of four plane curves of order four situated in the faces of the tetrahedron $R\,S\,T\,H$, and O, as a point of this double curve, must thus lie in *one* of those faces, namely in the polar plane $R\,S\,T$ of H, because the points O_1 and O_2, whose tangents intersect each other in O, lie on a straight line through H. The cone $[T]$ intersects τ in a conic k^2_3, likewise touching in O_1 and O_2 the lines OO_1, OO_2; the cone $[H]$ on the contrary has with τ only the line O_1O_2 counting double in common.

2. If we project r^4 out of O on an arbitrary plane π, then the projection is a plane curve k^4 with two cusps in the points of intersection of this plane with OO_1, OO_2; it is convenient to take for this plane of projection the polar plane of O with respect to the cone $[H]$, because then O_1, O_2, together with two other important points — of which we shall soon hear more — coïncide with their projections; the cuspidal tangents are nothing but the traces of the osculating planes of r^4 in O_1, O_2 with π.

The plane π intersects the cone $[H]$ in two generatrices; one is O_1O_2, the other intersects r^4 in two points D_1, D_2 coïnciding with their central projections on π, and in which k^4 touches the line D_1D_2, because the plane through this line and O is a tangential plane of $[H]$; so D_1D_2 is the double tangent of k^4, and H is the point of intersection of this double tangent with the connecting line O_1O_2 of the cusps [1]).

Each generatrix of $[H]$ contains two points of r^4, lying harmonically with respect to the point H and the point of intersection with the polar plane RST of H; so if we call h the line of intersection of this plane with π, it ensues immediately that *each line of π through H contains four points of k^4, lying harmonically in two pairs with respect to H and h*; each pair originates from two points on a generatrix of $[H]$.

If we consider O_1, O_2, D_1, D_2 as base-points of a pencil of conics, then for each curve of this pencil H is the pole of h; each curve containing the cusps and the points of contact of the double tangent of k^4, it cuts this curve in two more points P_1, P_2, whose connecting line passes through H. These pairs of points determine on k^4 a fundamental involution in such a way that on each ray through H lie two pairs, originating from the two pairs of points of r^4 on two generatrices of $[H]$ situated with O in one plane; the conics of the pencil are thus arranged by the rays out of H in pairs of a quadratic involution, whose double elements correspond to O_1O_2 and the double tangent d; the former consists of the conic of the pencil touching in O_1 and O_2 the cuspidal tangents, a curve which together with h forms the first polar curve of H with respect to k^4; the second must break up into the lines O_1O_2 and d, because this conic must touch the line d in D_1 and D_2. By the pencil (H) and the pencil of conics (O_1, O_2, D_1, D_2) paired involutorily conjugated to it k^4 is generated as the locus of the points of intersection of corresponding elements, where besides k^4 also the line d appears. [2]).

However, k^4 can be generated in still another way. Let us imagine through O_1O_2 instead of π another plane; this will intersect r^4 besides in O_1, O_2 in two more points P_1, P_2, whose connecting line passes through H and is divided harmonically by these three points and the plane RST; so the central projections P'_1, P'_2 are situated likewise on a line through H, and lie harmonically with respect to H and h. Let us now consider the pencil of conics (O_1, O_2, P'_1, P'_2). The different conics of this pencil are likewise involutorily paired by the pencil (H); the branchrays are again O_1O_2 and d, the double conics conjugated to them are the conic of the pencil touching in

[1]) See the paper of Prof. Jan de Vries (Proceedings of Amsterdam of Dec. 1908 p. 499): "On bicuspidal curves of order four".

[2]) J. de Vries, l. c. p. 500.

O_1, O_2 the cuspidal tangents and the one passing through D_1, D_2. The locus of the points of intersection of corresponding elements consists of k^4 and the line $P'_1 P'_2$.

We can finally allow P'_1, P'_2 to coïncide with the cusps, we can thus consider all the conics touching the cuspidal tangents in O_1, O_2; these too are paired involutorily by the pencil (H) whilst the branch-rays are again represented by d and $O_1 O_2$; to d is again conjugated the polar conic of H, to $O_1 O_2$ a conic having in each of the two cusps four coïnciding points with k^4 in common; so it can be nothing but the line $O_1 O_2$ counted double. As the locus of the points of intersection of corresponding elements of both pencils appears this time besides k^4 still the line $O_1 O_2$.

3. Among the planes of the pencil $(O_1 O_2)$ are of special importance those containing the cone-vertices R, S, or T; the former e.g. is the polar plane of O with respect to the cone $[R]$ and contains therefore the two generatrices RO_1, RO_2. Each of these cuts r^4 in one point more, e g. R_1 and R_2; the tangential planes along these generatrices to $[R]$ pass through O however; and from this ensues that the central projections of RO_1 and RO_2, cutting each other in the projection R' of R lying on h, must touch k^4 at the points R'_1, R'_2. We can, however, also bring through O two tangential planes to the cones $[S]$ and $[T]$, so: *out of each of the two cusps three tangents can be drawn to* k^4, *the traces of the tangential planes through* O *to the cones* $[R]$, $[S]$, $[T]$; *these tangents intersect each other in pairs in three points* R', S', T' *of* h [1]) (this also follows from the harmonic position of the whole figure with respect to H and h), *the projections of the three cone-vertices* R, S. T.

R e m a r k. If we take for k_1^2, k_2^2 (see § 1) two concentric circles, and if we then project the figure in space on a plane π (§ 2) which is parallel to τ, the oval of DESCARTES is generated and R', S', T' pass into the foci.

The twisted curve r^4 can be generated in six different ways as intersection of two of the four cones; let us take in particular $[R$ and $[H]$. If we make to pass through the line RH two planes μ_1, μ_2, harmonically separated by the planes RHS and RHT, the points of the two quadruples of points lying in these planes are situated two by two on four straight lines through each of the four cone-vertices; we now take for μ_1 the plane passing through the points O_1, O_2, R_1, R_2, and we shall call the four points in the other plane S_1, S_2, T_1, T_2. The

1) J. DE VRIES, l. c. p. 501.

latter four points lie thus with the former on four straight lines through S and on four others through T; let us now suppose e. g. that O_1S_1 passes through S. The tangential plane along the line SS_1O_1 to the cone $[S]$ has then as trace with τ the tangent in O_1 to k_2^2, i. e. the line OO_1, and from this ensues that the four points S_1, S_2, T_1, T_2 are nothing but the points of contact of the tangential planes through O of the cones $[S]$ and $[T]$ with r^4; however, we know now that these four points lie on two lines through R, so *in projection the points of contact S'_1, S'_2, T'_1, T'_2 of the tangents out of the cusps not passing through R' lie on two straight lines through R', namely S'_1 with T'_2, and S'_2 with T'_1.* [1]) That they lie also on two lines through H follows moreover again from the harmonic position with respect to H and h.

The polar plane of R with respect to the remaining three cones is simply the plane STH; it intersects the plane RST in the line ST (whose central projection falls on h), and the two lines of intersection with μ_1 and μ_2 lying in this plane in two points R^*, R^{**}, lying on ST, and situated harmonically with respect to S and T; R^* is then the point of intersection of O_1R_2, O_2R_1, R^{**} that of S_1T_1 and S_2T_2. If we transfer these results to the projection, we find: *of the complete quadrangle $O_1O_2R'_1R'_2$ two of the diagonal points are R', $R^{*'}$ (the third is H), and of the quadrangle $S'_1T'_2T'_1S'_2$ likewise R', $R^{**'}$; the points $R^{*'}$ and $R^{**'}$ lie harmonically with respect to S' and T'.* A similar property of the same points holds if we combine S'_1, S'_2 with the cusps and then regard the other four points; or finally if we couple T'_1, T'_2 to O_1, O_2 and join the other four to a complete quadrangle. And finally all six points $R'_1 \dots T'_2$ lie on a conic in consequence of the harmonic position with respect to h and H, whilst the points of intersection not lying on h of the six tangents out of the cusps lie likewise on a conic and at the same time in pairs on three straight lines through H.

4. Besides the group of 8 points just considered consisting of the two cusps and the points of contact of the six tangents passing through these points, there lie on k^4 still an infinite number of other such-like groups which have with respect to R', S', T', H the same properties. Let us for instance suppose that the plane μ_1 (§ 3) is still passing through RH but for the rest arbitrary, and let us suppose μ_2 again as harmonically conjugate to μ_1 with respect to HRS, HRT, then on r^4 a new group of 8 points is generated

[1]) J. DE VRIES, l. c. p. 498.

lying in pairs on four lines through each of the 4 vertices, and whose central projections thus possess the same property with respect to the points R', S', T', H. If we divide the 8 projections into two quadruples, in such a way that one belongs to the four points of μ_1, the other to those of μ_2, then the two quadruples form two complete quadrangles with the common diagonal points R' and H, whilst the others, $R^{*\prime}$ and $R^{**\prime}$, lie harmonically with respect to S' and T'; *the pairs of points* $R^{*\prime}$, $R^{**\prime}$ *on* h *form therefore a quadratic involution with the double points* S', T'. Similair properties hold for the two other possible divisions of the group of 8 points into two quadruples, namely with respect to the points S and T.

A special group of 8 points is found by choosing for the two planes μ the tangential planes through the line RH to the cone $[H]$, for these are likewise harmonically separated by HRS, HRT, but they furnish instead of 8 points 4 pairs of coïnciding points of r^4, namely the points of contact of the 4 tangents out of R to r^4. These points of contact lie in the polar plane ϱ (§ 3) of R and on two generatrices of the cone $[S]$, and likewise on two of the cone $[T]$; the tangents themselves pass in projection into the four tangents of k^4 through R' not passing through the cusps, so: *the points of contact of the four tangents of* k^4 *through* R' *not passing through the cusps are the vertices of a complete quadrangle whose diagonal-points are the points* S' T', H; the corresponding points $R^{*\prime}$, $R^{**\prime}$ are the points of intersection of the two sides of that quadrangle passing through H with h.

Another special group of 8 points is generated if we choose for the planes μ the tangential planes through RH to the cone $[R]$; we then find the four points of r^4 in the plane RST, and therefore in projection the points of intersection of r^4 with h, whose tangents indeed pass through H, in consequence of the harmonic position of k^4 with respect to h and H.

5. A group of 8 points of k^4 must be determined by *one* of these points; for the connecting line of this point with O intersects r^4 in *one* point, which determines with the line HR the plane μ_1; and by μ_1 at the same time μ_2 is determined. Planimetrically we can deduce out of *one* point of a group the other ones with the aid of the following property. The cone $[R]$ intersects the plane $\varrho = STH$ in a conic r^2, and we find that r^4 lies harmonically with respect to this and the point R, in that sense that the two points of r^4 on a generatrix of $[R]$ are always harmonically separated by R and the point of intersection with r^2; in particular r^2 contains

the four points of intersection of r^4 with ϱ, whose tangents pass through R, and also the points harmonically conjugate to R with respect to the pairs O_1, R_1 and O_2, R_2 lying in the polar plane $O_1 O_2 R$ of O with respect to the cone $[R]$.

By passing to the projection we find out of this the following property of k^4: *through the points of contact of the four tangents out of R' not passing through the cusps, a conic r'^2 can be brought touching the two tangents which do pass through the cusps in the points harmonically conjugate to R' with respect to the cusps and the corresponding points of contact; k^4 now lies harmonically with respect to R' and r'^2 in such a sense that the four points of intersection of k^4 with an arbitrary ray through R' arrange themselves into two pairs, each lying harmonically with R' and one of the points of intersection of that ray with r'^2.*

Of course also the points S' and T' possess such a conic, but the point H likewise, namely the line h counted double; for, the polar plane of H with respect to r^4 is the plane $R\,S\,T$, and the section with the cone $[H]$ lying in this plane is projected out of O into the line h to be counted double.

With the aid of the conic r'^2 it is easy to deduce out of *one* point of a group the seven other ones. If point 1 is taken arbitrarily, we find four others by the harmonic position of k^4 with respect to the points R', S', T', H and the corresponding conics; on the line $R'1$ e.g. are lying besides 1 still 3 points of k^4, but among these is only *one* forming with 1, R', and *one* of the points of intersection of $R'1$ with r'^2 a harmonic group. The three then still missing points we find by simply combining in proper fashion the already found one with R', S', T', or H.

Just as in § 2 by rays out of H now, too, a fundamental involution is generated on k^4 by those out of R' (or S', or T') in such a manner that on each ray lie two pairs, originating from the two pairs of points of r^4 on two generatrices of the cone $[R]$ lying with O in one plane; each pair is harmonically separated by R' and *one* of the two points of intersection of the indicated ray with r'^2. If now in two points of r^4 lying on a straight line through R we draw tangents, then these intersect each other in a point of the plane STH, and the locus of this point of intersection is a plane curve of order four, double curve of the developable of r^4, with double points in S, T, H and having with r^4 the four points of intersection of r^4 with the plane STH in common.

1). J. DE VRIES, l. c. p. 498.

Let us consider in particular the points O_1, R_1 (§ 3), lying on a straight line through R, and let us remember that the tangent in O_1 to r^4 passes through O; we shall then find on OO_1 a point of the double curve of order four, whilst the tangent in that point is the line of intersection of the osculating planes of r^4 in O_1 and R_1; that one of O_1 passes through O and furnishes in projection the cuspidal tangent in O_1 to k^4; so the central projection of the double curve will contain the cusps of k^4, and will touch the cuspidal tangents here.

So, summing up we find: *the point of intersection of the tangents to k^4 in two conjugate points of the involution generated by the pencil (R') generates a curve of order four with double points in S', T', H, situated harmonically with respect to h and H and having with k^4 in common the tangential points of the four tangents out of R' not passing through the cusps and likewise the cusps and the tangents in these.*

The new curve of order four cuts k^4 besides in the cusps (counting together for six points of intersection, and the tangential points of the four tangents out of R' not passing through the cusps, in six points more, of course again situated two by two harmonically with respect to h and H; now these six points lie twice with two of the four above mentioned points of contact (those namely whose connecting line passes through H) on a conic. For, the conic through two of those points of contact touching in O_1 and O_2 the cuspidal tangents of k^4, contains $2 \times 3 + 2 = 8$ points of intersection of the two curves of order four; so the other eight must also lie on a conic.

Also to the points S' and T' belongs such a locus of order four; as to the point H we can observe that in space the locus belonging to H lies in the plane RST and passes through O, so it passes by central projection into the line h counted three times, because each line passing through O and lying in the plane RST intersects the curve besides in O in three more points; indeed, through each point of h pass three pairs of tangents to k^4, whose chords of contact pass through H. The central projection of O itself however is undetermined, and so h is covered with points for the fourth time; the locus belonging to H consists thus of the line h counted four times.

R e m a r k. The properties found in this § hold in a somewhat more general form also for k^4 with two nodes, because they have been simply arrived at by centrally projecting the complete figure of the tetrahedron of Poncelet; the points R', S', T', H have then however not such a simple position as for the bicuspidal k^4.

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6. If we bring the vertex S of the cone $[S]$ on the surface of $[R]$, then T coïncides with S, whilst the point $S = T$ becomes a node of r^4; so k^4 possesses besides the two cusps a node and in this point S' and T' lie united. Out of each cusp only one tangent more can be drawn to k^4, and these two tangents intersect each other in the point R' lying on h. Through this point pass two more tangents to k^4, the projections of the two generatrices of the cone $[R]$ touching r^4. The points of contact of these two tangents with r^4 lie in the plane ϱ (§ 4), the common polar plane of R with respect to the cones $[S]$ and $[H]$, and so on the conic r^2 which has this plane in common with the cone $[R]$. This conic contains the vertex S; *so now k^4 is harmonically situated with respect to the point R' and the conic r'^2 passing through the double point and the points of contact of the two tangents out of R' not passing through the cusps, and touching the tangents out of R' which do pass through the cusps in the harmonically conjugated points of R' with respect to the cusps and the points of contact.*

If we bring through the line HR an arbitrary plane μ_1 (see § 4), then the harmonically conjugate μ_2 always coïncides with the tangential plane HRS to $[H]$, which plane contains no other point of r^4 than the node; of each group considered in § 4 of 8 points there are four coïnciding in the node, whilst the four remaining ones form a complete quadrangle with the diagonal points R, H, R^*. The tangents to r^4 in two points on a straight line through R intersect each other in ϱ and the locus of this point of intersection is a plane k^3 containing the node of r^4, having in this point a cusp (with cuspidal tangent in the plane RST), passing through H and having with r^4 two points in common, whose tangents pass through R. So if we pass to the central projection of r^4, we find that *by the rays of the pencil (R') on k^4 again a quadratic involution is generated in such a way that the two points of each pair form with R' and one of the two points of intersection of the ray under discussion with r'^2 a harmonic group; the point of intersection of the tangents in the points of a pair moves along a cubic curve lying harmonically with respect to h and H, containing the cusps of k^4 and touching here the cuspidal tangents, passing through the points of contact of the two tangents out of R' not passing through the cusps, passing through H and having in the node of k^4 a cusp with cuspidal tangent h.*

The cusps, the point of contact of the two tangents out of R' and the node represent all the twelve points of intersection of the cubic curve with k^4.

7. If, still imagining that the cone-vertices S and T are united in a point of the surface of $[H]$, we suppose the cone $[S]$ to be such that it touches the tangential plane in S to $[H]$, then r^4 gets a cusp in S and R coïncides with S and T, so that through r^4 pass but two quadratic cones; the plane $R\,S\,T$ remains determined, as the polar plane of H with respect to the cone $[R] = [S] = [T]$, and in this plane lies the tangent in the cusp R. The central projection now becomes a k^4 with three cusps, and the cuspidal tangent in R' is the line h. The double tangent through H remains determined as a trace of the plane of projection with the second tangential plane through O to $[H]$; however, as is easy to prove from the stereometric diagram, the points of contact must necessarily be imaginary.

As k^4 is situated harmonically with respect to h and H, the tangents in the cusps O_1, O_2 must intersect each other on h; however, h is the tangent in the cusp R': *so the three cuspidal tangents pass through one point.*

The point H forms with the cusps O_1, O_2, and the point of intersection of the line O_1O_2 with h a harmonic group; but now the same must hold for the two other sides of the $\Delta O_1O_2R'$ of the cusps; *the double tangent d is therefore the so-called harmonical line of the points of intersection of the three cuspidal tangents with respect to the triangle of the three cusps; and the points of contact of the double tangent are the nodes of the elliptic involution on d, of which the points of intersection of d with the sides of that triangle and with the cuspidal tangents in the opposite vertices are three pairs. With respect to each cuspidal tangent and the point of intersection of the opposite side of the triangle with it the curve lies harmonically with itself.*

Zoology. — "*On Ptilocodium repens a new Gymnoblastic Hydroid epizoic on a Pennatulid.*" By Miss WINIFRED E. COWARD B.Sc. Victoria University of Manchester. (Communicated by Prof. MAX WEBER).

(Communicated in the meeting of January 30, 1909.)

Order. Gymnoblastea — Anthomedusae.
Family. Ptilocodiidae. fam. nov.
Ptilocodium repens: gen. nov., sp. nov.
"Siboga" Expedition Stat. 289. 9° 0,3 S. 126° 24,5 E. 112 metres.

Among the Pennatulids sent to Professor HICKSON from the Siboga