Mathematics. — *"An integral-theorem of* GEGENBAUER." By Prof. KLUYVER.

(Communicated in the meeting of February 27, 1909).

GEGENBAUER has proved a theorem according to which the product of two functions of BESSEL $J^{\nu}(ax)$ and $J^{\nu}(bx)$ with the same parameter $\nu > -\frac{1}{2}$ can be given the form of a definite integral [1]).

In a former communication [2]) I have applied this theorem for the case $\nu = 0$ when reducing some discontinuous integrals containing functions of BESSEL. I shall now give in the following a direct proof of the indicated theorem and shall use it to extend former results.

1. In order to find the product of two functions of BESSEL

$$J^{\nu}(ax) = \left(\frac{ax}{2}\right)^{\nu} \sum_{h=0}^{h=\infty} \frac{(-1)^h \left(\frac{ax}{2}\right)^{2h}}{h!\,\Gamma(\nu+h+1)},$$

$$J^{\nu}(bx) = \left(\frac{bx}{2}\right)^{\nu} \sum_{h=0}^{h=\infty} \frac{(-1)^h \left(\frac{bx}{2}\right)^{2h}}{h!\,\Gamma(\nu+h+1)}$$

we can multiply the absolutely converging power series. It is then evident, that (supposing $b < a$) we find for the coefficient of an arbitrary power of x a finite hypergeometic series with the fourth argument $\frac{b^2}{a^2}$.

We get [3])

$$J^{\nu}(ax)\,J^{\nu}(bx) = \left(\frac{abx^2}{4}\right)^{\nu} \sum_{h=0}^{h=\infty} \frac{(-1)^h \left(\frac{x}{2}\right)^{2h}}{h!\,\Gamma(\nu+1)\Gamma(\nu+h+1)} a^{2h} F\left(-\nu-h, -h, \nu+1, \frac{b^2}{a^2}\right).$$

To transform the hypergeometric series appearing here I shall use the notation of RIEMANN for the general hypergeometric function

$$P \left| \begin{matrix} a & b & c & \\ \alpha & \beta & \gamma, & z \\ \alpha' & \beta' & \gamma' & \end{matrix} \right|,$$

$$(\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1)$$

$$P_a^{\alpha} = (z-a)^{\alpha}\,[1 + A_1(z-a) + A_2(z-a)^2 + \ldots\ .].$$

[1]) NIELSEN. Handbuch der Theorie der Cylinderfunktionen, page 182.

[2]) Proceedings 1905.

[3]) NIELSEN. page 20.

It is then evident that for two of the singular points the differences of the exponents are equal, so that besides the ordinary substitution of order one also substitutions of order two as

$$P_1^{\beta}\left|\begin{matrix} 0 & -1 & +1 & \\ \alpha & \beta & \beta, & z \\ \alpha' & \beta' & \beta' & \end{matrix}\right| = 2^{-\beta} P_1^{\frac{\beta}{2}}\left|\begin{matrix} 0 & \infty & 1 & \\ \frac{\alpha}{2} & 0 & \beta, & z^2 \\ \frac{\alpha'}{2} & \frac{1}{2} & \beta' & \end{matrix}\right|$$

and

$$P_0^{\alpha}\left|\begin{matrix} 0 & -1 & +1 & \\ \alpha & \beta & \beta, & z \\ \alpha' & \beta' & \beta' & \end{matrix}\right| = P_0^{\frac{\alpha}{2}}\left|\begin{matrix} 0 & \infty & 1 & \\ \frac{\alpha}{2} & 0 & \beta, & z^2 \\ \frac{\alpha'}{2} & \frac{1}{2} & \beta' & \end{matrix}\right|$$

are possible.

The indicated reduction runs as follows:

$$F\left(-h, -v-h, v+1, \frac{b^2}{a^2}\right) = P_0^0\left|\begin{matrix} 0 & \infty & 0 & \\ 0 & -h & 1 & , \frac{b^2}{a^2} \\ -v & -v-h & 2v+1+2h & \end{matrix}\right| =$$

$$= \left(\frac{b}{a}\right)^h P_0^{-\frac{h}{2}}\left|\begin{matrix} 0 & \infty & 1 & \\ -\frac{h}{2} & -\frac{h}{2} & 0 & , \frac{b^2}{a^2} \\ -v-\frac{h}{2} & -v-\frac{h}{2} & 2v+1+2h & \end{matrix}\right| = .$$

$$= 2^{\frac{h}{2}}\left(\frac{b}{a}\right)^h P_1^{-\frac{h}{2}}\left|\begin{matrix} 0 & -1 & +1 & \\ 0 & -\frac{h}{2} & -\frac{h}{2}, & \frac{a^2-b^2}{a^2+b^2} \\ 2v+1+2h & -v-\frac{h}{2} & -v-\frac{h}{2} & \end{matrix}\right| =$$

$$= 2^h\left(\frac{b}{a}\right)^h P_1^{-\frac{h}{2}}\left|\begin{matrix} 0 & \infty & 1 & \\ 0 & 0 & -\frac{h}{2} & , \left(\frac{a^2-b^2}{a^2+b^2}\right)^2 \\ v+\frac{1}{2}+h & \frac{1}{2} & -v-\frac{h}{2} & \end{matrix}\right| =$$

$$= 2^h \left(\frac{b}{a}\right)^h P_0^{\frac{h}{2}} \left| \begin{matrix} 0 & \infty & 1 & \\ -\frac{h}{2} & 0 & 0 & , \frac{4a^2b^2}{(a^2+b^2)^2} \\ -v-\frac{h}{2} & \frac{1}{2} & v+\frac{1}{2}+h & \end{matrix} \right| =$$

$$= 2^h \left(\frac{b}{a}\right)^h P_0^{-h} \left| \begin{matrix} 0 & -1 & +1 & \\ -h & 0 & 0 & , \frac{2ab}{a^2+b^2} \\ -2v-h & v+\frac{1}{2}+h & v+\frac{1}{2}+h & \end{matrix} \right| =$$

$$= \left(\frac{a+b}{a}\right)^{2h} P_0^0 \left| \begin{matrix} 0 & -1 & 1 & \\ 0 & -h & 0 & , \frac{2ab}{a^2+b^2} \\ -2v & v+\frac{1}{2} & v+\frac{1}{2}+h & \end{matrix} \right| =$$

$$= \left(\frac{a+b}{a}\right)^{2h} P_0^0 \left| \begin{matrix} 0 & \infty & 1 & \\ 0 & -h & 0 & , \frac{4ab}{(a+b)^2} \\ -2v & v+\frac{1}{2} & v+\frac{1}{2}+h & \end{matrix} \right| =$$

$$= \left(\frac{a}{a+b}\right)^{2h} F\left(-h,\ v+\tfrac{1}{2},\ 2v+1,\ \frac{4ab}{(a+b)^2}\right).$$

So finally we have

$$a^{2h} F\left(-h,\ -v-h,\ v+1,\ \frac{b^2}{a^2}\right) = (a+b)^{2h} F\left(-h,\ v+\tfrac{1}{2},\ 2v+1,\ \frac{4ab}{(a+b)^2}\right),$$

a transformation given by GAUSS.

We now substitute for the just given hypergeometrical series the integral

$$\frac{2^{2v}\,\Gamma(v+1)}{\sqrt{\pi}\,\Gamma(v+\frac{1}{2})} \int_0^1 z^{v-\frac{1}{2}} (1-z)^{v-\frac{1}{2}} \left(1 - \frac{4ab}{(a+b)^2} z\right)^h dz,$$

and we find if in the integral we put $z = \cos^2 \frac{\varphi}{2}$

$$a^{2h} F\left(-h,\ -v-h,\ v+1,\ \frac{b^2}{a^2}\right) = \frac{\Gamma(v+1)}{\sqrt{\pi}\,\Gamma(v+\frac{1}{2})} \int_0^\pi (a^2+b^2-2ab\cos\varphi)^h \sin^{2v}\varphi\, d\varphi.$$

In the development found for the product $J^v(ax)\ J^v(bx)$ the above mentioned integral is introduced.

Putting

$$a^2 + b^2 - 2ab\cos\varphi = \Omega^2,$$

we obtain

51*

$$J^\nu(ax)\,J^\nu bx = \frac{a^\nu\, b^\nu\, x^\nu}{2^\nu\sqrt{\pi}\;\Gamma(\nu+\frac{1}{2})}\int\limits_0^\pi sin^{2\nu}\,\varphi\, d\varphi \left(\frac{x}{2}\right)^\nu \sum_{h=0}^{h=\infty} \frac{(-1)^h \left(\frac{\Omega x}{2}\right)^{2h}}{h!\;\Gamma(\nu+h+1)},$$

or

$$J^\nu(ax)\,J^\nu(bx) = \frac{a^\nu\, b^\nu\, x^\nu}{2^\nu\sqrt{\pi}\;\Gamma(\nu+\frac{1}{2})}\int\limits_0^\pi \frac{J^\nu(\Omega x)}{\Omega^\nu}\, sin^{2\nu}\,\varphi\, d\varphi,$$

by which the indicated theorem of GEGENBAUER has been proved.

2. With the aid of this theorem we can extend some well-known results concerning discontinuous integrals, in which functions of BESSEL appear; particularly do the two following theorems [1])

$$\frac{c^{\nu+1}}{a^\nu}\int\limits_0^\infty J^{\nu+1}(uc)\,J^\nu(ua)\,du = \begin{cases} 1 & for\ a < c, \\ 0 & for\ a > c, \end{cases}$$

$$\frac{c^{\nu+2}}{a^\nu}\int\limits_0^\infty J^{\nu+2}(uc)\,J^\nu(ua)\,\frac{du}{v} = \begin{cases} \frac{1}{2}(c^2-a^2) & for\ a < c, \\ 0 & for\ a > c. \end{cases}$$

lend themselves to this extension.

The theorem of GEGENBAUER namely allows on the ground of these results to determine in certain supposition the value of the discontinuous integrals

$$W_1 = \frac{c^{\nu+1}}{a_1^{\,\nu} a_2^{\,\nu} \ldots a_n^{\,\nu}}\int\limits_0^\infty J^{\nu+1}(uc)\,J^\nu(ua_1)\,J^\nu(ua_2)\ldots J^\nu(ua_n)\,\frac{du}{u^{\nu(n-1)}},$$

$$W_2 = \frac{c^{\nu+2}}{a_1^{\,\nu} a_2^{\,\nu} \ldots a_n^{\,\nu}}\int\limits_0^\infty J^{\nu+2}(uc)\,J^\nu(ua_1)\,J^\nu(ua_2)\ldots J^\nu(ua_n)\,\frac{du}{n^{\nu(n-1)+1}},$$

in which the number of J-functions is arbitrary and ν is $> -\frac{1}{2}$.

Let us think the positive numbers a_1, a_2, ... a_n to be successive sides of a broken line $OA_1A_2 \ldots A_n$, let us put $\angle\, OA_kA_{k+1} = \varphi_k$ and $OA_k = s_k$, then we find successively

$$\frac{1}{a_1^{\,\nu} a_2^{\,\nu} u^\nu} J^\nu(ua_1)\,J^\nu(ua_2) = \frac{1}{2^\nu\sqrt{\pi}\;\Gamma(\nu+\frac{1}{2})}\int\limits_0^\pi \frac{J^\nu(us_2)}{s_2^{\,\nu}}\, sin^{2\nu}\,\varphi_1\, d\varphi_1,$$

$$\frac{1}{s_2^{\,\nu} a_3^{\,\nu} u^\nu} J^\nu(us_2)\,J^\nu(ua_3) = \frac{1}{2^\nu\sqrt{\pi}\;\Gamma(\nu+\frac{1}{2})}\int\limits_0^\pi \frac{J^\nu(us_3)}{s_3^{\,\nu}}\, sin^{2\nu}\,\varphi_2\, d\varphi_2,$$

. . . .

[1]) NIELSEN, p. 198.

$$\frac{1}{s^{\nu}_{n-1}a^{\nu}_{n}u^{\nu}} J^{\nu}(us_{n-1})\, J^{\nu}(ua_n) = \frac{1}{2^{\nu}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^{\pi} \frac{J^{\nu}(us_n)}{s^{\nu}_n} \sin^{2\nu} \varphi_{n-1}\, d\varphi_{n-1}\,,$$

so that we get

$$W_1 = \frac{1}{[2^{\nu}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})]^{n-1}} \int_0^{\pi} \sin^{2\nu}\varphi_1\, d\varphi_1 \int_0^{\pi} \sin^{2\nu}\varphi_2\, d\varphi_2 \ldots \int_0^{\pi} \sin^{2\nu}\varphi_{n-1}\, d\varphi_{n-1} \times$$

$$\times \frac{c^{\nu+1}}{s^{\nu}_n} \int_0^{\infty} J^{\nu+1}(uc)\, J^{\nu}(us_n)\, du,$$

$$W_2 = \frac{1}{[2^{\nu}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})]^{n-1}} \int_0^{\pi} \sin^{2\nu}\varphi_1\, d\varphi_1 \int_0^{\pi} \sin^{2\nu}\varphi_2\, d\varphi_2 \ldots \int_0^{\pi} \sin^{2\nu}\varphi_{n-1}\, d\varphi_{n-1} \times$$

$$\times \frac{c^{\nu+2}}{s^{\nu}_n} \int_0^{\infty} J^{\nu+2}(uc)\, J^{\nu}(us_n)\, \frac{du}{u}.$$

Let now in the first place be

$$c > a_1 + a_2 + \ldots + a_n\,,$$

then c is certainly greater than s_n, and we find

$$W_1 = \left[\frac{1}{2^{\nu}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^{\pi} \sin^{2\nu} \alpha\, d\alpha\right]^{n-1},$$

$$W_2 = \tfrac{1}{2}\,(c^2 - a_1^{\,2} - a_2^{\,2} \ldots - a^2_n) \left[\frac{1}{2^{\nu}\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^{\pi} \sin^{2\nu} \alpha\, d\alpha\right]^{n-1}.$$

As

$$\int_0^{\pi} \sin^{2\nu} \alpha\, d\alpha = \frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}$$

we find as final result

$$W_1 = \frac{c^{\nu+1}}{a^{\nu}_1 a^{\nu}_2 \ldots a^{\nu}_n} \int_0^{\infty} J^{\nu+1}(uc)\, J^{\nu}(ua_1)\, J^{\nu}(ua_2) \ldots J^{\nu}(ua_n)\, \frac{du}{u^{\nu(n+1)}} = \frac{1}{[2^{\nu}\,\Gamma(1+\nu)]^{n-1}},$$

$$W_2 = \frac{c}{a^{\nu}_1 a^{\nu}_2 \ldots a^{\nu}_n} \int_0^{\infty} J^{\nu+2}(uc) J^{\nu}(ua_1) J^{\nu}(ua_2) \ldots J^{\nu}(ua_n)\, \frac{du}{u^{\nu(n-1)+1}} = \frac{c^2 - a_1^{\,2} - a_2^{\,2} \ldots - a_n^{\,2}}{2[2^{\nu}\,\Gamma(1+\nu)]^{n-1}}.$$

$$(c > a + a + \ldots + a_n).$$

Still in a second case the values of the integrals W_1 and W_2 are known. Let a_1 exceed all other numbers a and let us put

$$a_1 > c + a_2 + \ldots + a_n.$$

It is necessary then for all values of $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ that the closing side of the broken line be greater than c and taking into consideration the two integral-theorems, serving as starting point, we conclude that the integrals W_1 and W_2 have both become zero.

3. We might ask whether results as arrived at above are also attained when the functions $J(ua)$ behind the sign of integration have not all the same parameter. The following operation shows that this is partly the case. Supposing that $\mu_1, \mu_2, \ldots, \mu_n$ are numbers greater than ν, then under a definite condition the evaluation of the integrals

$$W_3 = \frac{c^{\nu+1}}{a_1^{\mu_1} a_2^{\mu_2} \ldots a_n^{\mu_n}} \int_0^\infty J^{\nu+1}(uc)\, J^{\mu_1}(ua_1)\, J^{\mu_2}(ua_2) \ldots J^{\mu_n}(ua_n) \frac{du}{u^{\Sigma\mu-\nu}},$$

$$W_4 = \frac{c^{\nu+2}}{a_1^{\mu_1} a_2^{\mu_2} \ldots a_n^{\mu_n}} \int_0^\infty J^{\nu+2}(uc)\, J^{\mu_1}(ua_1)\, J^{\mu_2}(ua_2) \ldots J^{\mu_n}(ua_n) \frac{du}{u^{\Sigma\mu-\nu+1}}$$

can be reduced to that of the integrals W_1 and W_2.

For the reduction of W_3 and W_4 we can repeatedly apply the formula [1])

$$J^{\mu}(ua) = \frac{(ua)^{\mu-\nu}}{2^{\mu-\nu-1}\Gamma(\mu-\nu)} \int_0^{\frac{\pi}{2}} J^{\nu}(ua\cos\alpha) \cos^{\nu+1}\alpha \sin^{2\mu-2\nu-1}\alpha \, d\alpha\,.$$

We obtain in this way

$$W_3 = \frac{1}{2^{\Sigma\mu-n\nu}} \cdot \frac{2}{\Gamma(\mu_1-\nu)} \int_0^{\frac{\pi}{2}} \cos^{2\nu+1}\alpha_1 \sin^{2\mu_1-2\nu-1}\alpha_1 d\alpha_1 \ldots$$

$$\ldots \frac{2}{\Gamma(\mu_n-\nu)} \int_0^{\frac{\pi}{2}} \cos^{2\nu+1}\alpha_n \sin^{2\mu_n-2\nu-1}\alpha_n d\alpha_n \times$$

$$\times \frac{c^{\nu+1}}{(a_1\cos\alpha_1)^{\nu} \ldots (a_n\cos\alpha_n)^{\nu}} \int_0^\infty J^{\nu+1}(uc)\, J^{\nu}(ua_1\cos\alpha_1) \ldots J^{\nu}(ua_n\cos\alpha_n) \frac{du}{u^{\nu(n-1)}},$$

$$W_4 = \frac{1}{2^{\Sigma\mu-n\nu}} \cdot \frac{2}{\Gamma(\mu_1-\nu)} \int_0^{\frac{\pi}{2}} \cos^{2\nu+1}\alpha_1 \sin^{2\mu_1-2\nu-1}\alpha_1 d\alpha_1 \ldots$$

$$\ldots \frac{2}{\Gamma(\mu_n-\nu)} \int_0^{\frac{\pi}{2}} \cos^{2\nu+1}\alpha_n \sin^{2\mu_n-2\nu-1}\alpha_n d\alpha_n \times$$

[1]) Nielsen, page 181.

$$\times \frac{c^{\nu+2}}{(a_1 \cos \alpha_1)^{\nu} \ldots (a_n \cos \alpha_n)^{\nu}} \int_0^{\infty} J^{\nu+2}(uc)\, J^{\nu}(ua_1 \cos \alpha_1) \ldots J^{\nu}(ua_n \cos \alpha_n) \frac{du}{u^{\nu(n-1)+1}}.$$

If now is given $c > a_1 + a_2 + \ldots + a_n$, then during the integration the inequality

$$c > a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + \ldots + a_n \cos \alpha_n$$

will continually hold and the results concerning the integrals W_1 and W_2 can be applied.

Remembering that we have

$$\frac{2}{\Gamma(\mu-\nu)} \int_0^{\frac{\pi}{2}} \cos^{2\nu+1} \alpha \sin^{2\mu-2\nu-1} \alpha d\alpha = \frac{\Gamma(1+\nu)}{\Gamma(1+\mu)},$$

$$\frac{2}{\Gamma(\mu-\nu)} \int_0^{\frac{\pi}{2}} \cos^{2\nu+3} \alpha \sin^{2\mu-2\nu-1} \alpha d\alpha = \frac{\Gamma(2+\nu)}{\Gamma(2+\nu)},$$

we finally find

$$W_3 = \frac{c^{\nu+1}}{a_1^{\mu_1} a_2^{\mu_2} \ldots a_n^{\mu_n}} \int_0^{\infty} J^{\nu+1}(uc)\, J^{\mu_1}(ua_1)\, J^{\mu_2}(ua_2) \ldots J^{\mu_n}(ua_n) \frac{du}{u^{\Sigma\mu-\nu}} =$$

$$= \frac{\Gamma(1+\nu)}{2^{\Sigma\mu-\nu}\, \Gamma(1+\mu_1)\Gamma(1+\mu_2) \ldots \Gamma(1+\mu_n)},$$

$$W_4 = \frac{c^{\nu+2}}{a_1^{\mu_1} a_2^{\mu_2} a_n^{\mu_n}} \int_0^{\infty} J^{\nu+2}(uc) J^{\mu_1}(ua_1) J^{\mu_2}(ua_2) \ldots J^{\mu_n}(ua_n) \frac{du}{u^{\Sigma\mu-\nu+1}} =$$

$$= \frac{\left(c^2 - \frac{\nu+1}{\mu_1+1} a_1^2 - \frac{\nu+1}{\mu_2+1} a_2^2 \ldots - \frac{\nu+1}{\mu_n+1} a_n^2\right) \Gamma(1+\nu)}{2^{\Sigma\mu-\nu+1}\, \Gamma(1+\mu_1)\, \Gamma(1+\mu_2) \ldots \Gamma(1+\mu_n)}.$$

$$(c > a_1 + a_2 + \ldots + a_n).$$

In particular we find out of the obtained value for W_3 for $\mu = \frac{1}{2}$, $\nu = -\frac{1}{2}$

$$\int_0^{\infty} \frac{\sin uc \sin ua_1 \sin ua_2 \ldots \sin ua_n}{u^{n+1}} du = \frac{\pi}{2} a_1 a_2 \ldots a_n,$$

for $\mu = -\frac{1}{2}$, $\nu = -\frac{1}{2}$

$$\int_0^{\infty} \frac{\sin uc \cos ua_1 \cos ua_2 \ldots \cos ua_n}{u} du = \frac{\pi}{2},$$