Mathematics. — "*A family of differential equations of the first order.*" By Prof. JAN DE VRIES.

(Communicated in the meeting of February 27, 1909).

1. The tangent in (x, y) to the integral curve of

$$\frac{dy}{dx} = P(x)\, y + Q(x)$$

is represented by

$$Y - y = \{P(x)\, y + Q(x)\}\,(X - x).$$

For the points of the line $x = m$ we have thus

$$\{Q_m (X - m) - Y\} + y\, \{P_m (X - m) + 1\} = 0.$$

The tangents indicated by them form therefore a pencil of rays having the point

$$x_0 = m - \frac{1}{P_m}, \quad y_0 = -\frac{Q_m}{P_m}$$

as vertex.

I call this point the *pole* of the line $x = m$.

By a projective transformation the linear equation is transformed into a differential equation, determining a pencil of rays of which each ray has a definite pole. Each line, connecting the vertex S of that pencil with a pole, having to touch in S an integral curve S is a *singular point*, i. e. a point where y' is indefinite. To confirm this we transform the linear equation by the substitution

$$x = \frac{a_1 u + a_2 v + a_3}{c_1 u + c_2 v + c_3} = \frac{\alpha}{\gamma}, \quad y = \frac{b_1 u + b_2 v + b_3}{c_1 u + c_2 v + c_3} = \frac{\beta}{\gamma}.$$

On account of that

$$y' = P(x)\, y + Q(x)$$

passes into

$$\frac{dv}{du} = \frac{(a_1\gamma - c_1\alpha)(\beta P^* + \gamma Q^*) - (b_1\gamma - c_1\beta)\gamma}{(b_2\gamma - c_2\beta)\,\gamma - (a_2\gamma - c_2\alpha)(\beta P^* + \gamma Q^*)}$$

where

$$P^* = P\left(\frac{\alpha}{\gamma}\right), \qquad Q^* = Q\left(\frac{\alpha}{\gamma}\right).$$

The pencil $x = m$ is then transformed into the pencil of rays

$$\alpha = m\gamma.$$

For $\alpha = 0,\ \gamma = 0$ we really find $\frac{dv}{du} = \frac{0}{0}$. [1])

[1]) Also the points $\beta = 0,\ \gamma = 0$ and $\gamma = 0,\ P^* = 0$ are singular.

A pencil of rays with the indicated property I call *critical.*

From the following ensues the property:

When a singular *point of a differential equation of the first order is the vertex of a* critical pencil of rays, *then this equation can be reduced by a projective transformation into a* linear *one.*

If $x = x_1$, $y = y_1$ is the vertex of a critical pencil, then the substitution

$$x - x_1 = \frac{1}{v} \quad , \quad y - y_1 = \frac{u}{v}$$

leads to the aim in view.

Example I.

The equation

$$\frac{dy}{dx} = \frac{x^3 + 2x^2y + xy^3 + y^4}{x^3 + x^2y^2 + xy^3}$$

has in $x = 0$, $y = 0$ a singular point.

The tangent in a point of the line $y = mx$ is indicated by

$$Y - mx = \frac{(m+1)\,m^3x + 2m + 1}{(m+1)\,m^2x + 1}(X - x).$$

By reduction it is evident that the parameter x appears here only linear; so the pencil is critical.

For the locus of the poles we find the cubic curve

$$x = -\frac{1}{(m+1)\,m^2}\ ,\ y = -\frac{2m+1}{(m+1)\,m^2}\,.$$

By the substitution

$$x = \frac{1}{v}\ ,\ y = \frac{u}{v}$$

$y = mx$ passes into $u = m$, and the given equation into

$$\frac{dv}{du} + \frac{v}{u+1} + u^2 = 0\,.$$

2. Let us treat in particular the equation

$$\frac{dy}{dx} = P(x)y\,.$$

Here the locus of the poles of $x = m$ is the line $y = 0$ (*polar line*).

As the homogeneous equation

$$\frac{d\eta}{d\xi} = f\left(\frac{\eta}{\xi}\right)$$

has the critical pencil of rays $\eta = m\xi$, with the line at infinity as *polar line*, the question arises whether it is possible to transform the homogeneous equation projectively into an equation of the form $y' = P(x)y$.

If we put

$$\xi = \frac{1}{y}\ , \ \eta = \frac{x}{y}\ ,$$

then the right line at infinite distance passes into $y = 0$, the pencil of rays $\eta = m\xi$ into the pencil $x = m$ and the differential equation into the divided equation

$$\frac{dy}{y} = \frac{dx}{x - f(x)}\,.$$

Also the equation

$$\frac{dy}{dx} = Q(x)$$

has for the critical pencil $x = m$ a *polar line*, namely the line at infinity. So here the vertex of the pencil lies on the polar line.

As we can always regulate a projectivity between two point-fields in such a way that a point and a line of the first field are conjugated to a definite point and a definite line of the second, the property holds:

If to a singular *point belongs a* polar line, *the differential equation can be transformed projectively into a divided equation of the form*

$$\frac{dy}{y} = P(x)\,dx,$$

unless the polar line *passes* through *the singular point. In this case we arrive by projective transformation at a divided equation of the form*

$$dy = Q(x)\,dx.$$

Example II.

The equation

$$\frac{dy}{dx} = \frac{y^2 - x}{xy + x}$$

has two critical pencils of rays:

$y = mx$ with the polar line $x + y = 0$,

$y + 1 = m(x - 1)$ with the polar line $x = 0$.

For the former pencil the vertex lies on the polar line. By the transformation

$$x = \frac{u}{v}\ , \ y = \frac{1-u}{v}\ , \ x + y = \frac{1}{v}$$

this line is thrown to infinity, whilst $y = mx$ passes into $u = 1 : (m+1)$. We find

$$\frac{dv}{du} = \frac{u-1}{u}, \quad u - v = lg\, u + C$$

and finally from this

$$\frac{x-1}{x+y} = lg \frac{x}{x+y} + C.$$

To make use of the second pencil we determine a projective transformation, which transforms $(y+1) : (x-1)$ into a linear function of u and $x = 0$ into $v = 0$. These conditions are satisfied by the substitution

$$x = \frac{v}{u+v}, \quad y = \frac{u-v+1}{u+v}, \quad \frac{y+1}{x-1} = -\frac{2u+1}{u}.$$

We now find

$$\frac{dv}{v} = \frac{u\, du}{(u+1)^2}.$$

3. When the linear equation

$$y' = P(x)y + Q(x)$$

has a *polar line* we can transform this projectively into the line at infinity. The *linear* equation is then transformed into a *homogeneous* one, or, where this is not possible into an equation of the form

$$\frac{dv}{du} = f(u).$$

In the latter case $u = m$ is the critical pencil; in the former where

$$\frac{dv}{du} = f\left(\frac{v}{u}\right)$$

the point $u = 0$, $v = 0$ is the vertex of the critical pencil of rays.

Let

$$ax + by + c = 0$$

be the polar line of

$$y' = P(x)y + Q(x),$$

thus the locus of the pole

$$x = m - \frac{1}{P_m}, \quad y = -\frac{Q_m}{P_m}.$$

If we put

$$m - \frac{1}{P_m} = f_m,$$

then

$$Q_m = \frac{af_m + c}{b(m - f_m)},$$

and the linear equation obtains the form

$$b(x - f(x))\, y' = by + af(x) + c.$$

It is clear, that by the substitution

$$ax + by + c = \frac{1}{u} \quad , \quad x = \frac{v}{u}$$

the polar line is brought to infinity and the point of intersection of the rays $x = m$ is brought into the origin.

After some reduction we find indeed the *homogeneous* equation

$$\frac{dv}{du} = f\left(\frac{v}{u}\right).$$

This reduction is apparently of no use when the polar line has as equation $x + c = 0$. Then

$$m - \frac{1}{P_m} = -c,$$

and the linear equation has this form:

$$y' = \frac{y}{x+c} + Q(x).$$

By the substitution

$$x + c = \frac{1}{u} \quad , \quad y = \frac{v}{u}$$

the polar line is transformed into the line at infinity, the pencil $x = m$ into the pencil $u = 1 : (m + c)$, and we find the divided equation

$$dv + Q\left(\frac{1}{u} - c\right)\frac{du}{u} = 0.$$

4. Let us look at a few more examples.

Example III. The equation

$$\frac{dy}{dx} = \frac{xy - y^2 + 2}{x^2 - xy - 2}$$

has singular points in $x = 1$, $y = -1$; $x = -1$, $y = 1$ and in the point at infinity on $x = y$.

The pencil $x - y = m$ is critical. We find for the tangent

$$(my + m^2 - 2)(Y - y) = (my + 2)(X - y - m),$$

and out of this the *polar line* $x + y = 0$.

By the transformation

$$x = v + u \quad , \quad y = v - u$$

the pencil $x - y = m$ passes into the pencil $2u = m$, the polar line into $v = 0$, and we find, as we ought to, the divided equation

$$\frac{dv}{v} = \frac{u\,du}{u^2-1}.$$

Finally we find as integral curves the conics

$$(x-y)^2 + C(x+y)^2 = 4,$$

touching one another in the singular points $x = \pm 1, y = \mp 1$. The point of intersection of the lines $x - y = m$ is the double point of the pair of lines $(v-y)^2 = 4$.

When applying the transformation $x + y = \frac{2}{u}$, $x-y = \frac{2v}{u}$ we find that $x-y = m$ passes into $v = \frac{1}{2} mu$, whilst the polar line is brought at infinity. We find then, as we ought to, a homogeneous equation, namely

$$\frac{dv}{du} = \frac{u}{v}.$$

For a ray of the pencil

$$y + 1 = m(x-1)$$

we find

$$y' = \frac{(x-1)\,y - (y-2)(y+1)}{(x-1)(x+2) - x(y+1)} = \frac{y-m(y-2)}{x+2-mx} = \frac{(m-1)\,mx + (1-2m-m^2)}{(m-1)\,x-2},$$

thus for the tangent

$$[(m-1)\,x-2]\;Y = [(m-1)\,mx + (1-2m-m^2)]\;X + 2\,(m+1).$$

Therefore this pencil is also critical. The poles lie on the line $y - x = 2$.

Example IV.

The equation

$$y' = \frac{xy-y^2+y}{x^2-xy-y+1}$$

has $x = 0$, $y = 1$ as singular point.

The pencil $y - 1 = mx$ is critical and has $y = 0$ as polar line. By the substitution $y - 1 = ux$,

$$x = -\frac{1}{u+v} \quad , \quad y = \frac{v}{u+v}$$

we find the divided equation

$$\frac{dv}{v}=\frac{u+1}{u^2+1}\,du.$$

EXAMPLE V.

The equation

$$(x^2-y)\,y'=xy-1$$

has a singular point in $x=1$, $y=1$, which is the vertex of a critical pencil with the polar line $x+y+1=0$.

By the substitution

$$x-1=\frac{3}{v-u-1}\quad,\quad y-1=\frac{3u}{v-u-1}$$

this pencil passes into $u=$ const., whilst the polar line is transformed into $v=0$. We then find

$$\frac{dv}{v}=\frac{u-2}{u^2-u+1}\,du$$

EXAMPLE VI.

The equation

$$xyy'=x+y^2$$

has the critical pencil $y=mx$ with the polar line $x=0$ passing through the vertex. In connection with this the substitution

$$x=\frac{1}{v}\quad,\quad y=\frac{u}{v}$$

furnishes the divided equation

$$udu+dv=0.$$

The integral curves

$$y^2+2x=Cx^2$$

are conics having in O a contact of four points with $x=0$ as tangent.

EXAMPLE VII. $$x^3\frac{dy}{dx}=x^2y+y^2.$$

This equation of BERNOULLI has a critical pencil $y=mx$ with the polar line $x=0$. By the substitution $x=1:v$, $y=u:v$ it is transformed into $u^{-2}du+dv=0$. Out of this we find $x^2-y+Cxy=0$.

5. When each ray through a singular point determines a system of tangents with *index two,* then the equation is projectively reducible to an equation of the form

$$\frac{dy}{dx}=\frac{N(x)y^2+P(x)y+Q(x)}{R(x)y+S(x)}.$$

For, this equation determines for $x=m$ the tangents of a conic and by the substitution

$$x=\frac{\alpha}{\gamma}\ , \qquad y=\frac{\beta}{\gamma}$$

(see § 1) it is transformed into an equation having in $\alpha=0, \gamma=0$ a singular point, whilst each ray of the pencil $\alpha=m\gamma$ possesses the above indicated property.

The equation

$$x\frac{dy}{dx}=\frac{x^3+y^3-2x^2y^2-xy}{y^2-2x^2y-x}$$

is in this case, for each ray $y=mx$ furnishes a system of tangents with index two. By the substitution

$$x=\frac{1}{v}\ , \qquad y=\frac{u}{v}$$

it passes into the equation (of RICCATI)

$$\frac{dv}{du}=2u-u^2v+v^2.$$

This can be reduced with the aid of the solution $v=u^2$ to the equation (of BERNOULLI)

$$\frac{dw}{du}=u^2w+w^2,$$

where $w=v-u^2$. By $w=z^{-1}$ we then arrive at a linear differential equation.

Botany. — Mr. VAN DER STOK presents in behalf of S. H. KOORDERS a communication entitled: "*Polyporandra Junghuhnii, a hitherto undescribed species of the order of Icacinaceae, found in 's Rijks Herbarium at Leiden by* S. H. KOORDERS" *(Plantae Junghuhnianae ineditae II)* [1]).

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Polyporandra Junghuhnii, KDS *n. spec. Frutex? scandens, ramulis teretiusculis novellis pubescentibus. Folia opposita, oblonga, basi acuta vel obtusa, apice sensim acuminata;* 12—13 *cm. longa et* 4—5 *cm. lata, petiolo* 1—1½ *cm. longo, subcoriacea, supra praeter costam*

[1]) Continuation of *Plantae Junghuhnianae ineditae I* in Proceedings of the Mathematical and Physical Section, of June 27 1908, p. 158—162.