**Mathemathics.** — *"On a class of differential equations of the first order and the first degree."* By Prof. W. KAPTEYN.

1. In the last meeting of this Academy Prof. J. DE VRIES gave a geometrical criterion for determining whether or not a given differential equation of the first order and the first degree may be reduced by a homographic substitution to a linear equation or to an equation of the form

$$\frac{dy}{dx}=\frac{N(x)y^2+P(x)y+Q(x)}{R(x)y+S(x)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The object of this paper is to examine the general form of all those equations which by a homographic substitution may be reduced to the equation (1). It is evident that this general form will give at the same time all the equations which are reducible either to the general equation of RICCATI, or to the linear form.

2. Let the substitution be

$$x=\frac{a_1u+a_2v+a_3}{c_1u+c_2v+c_3}=\frac{\alpha}{\gamma} \qquad y=\frac{b_1u+b_2v+b_3}{c_1u+c_2v+c_3}=\frac{\beta}{\gamma} \quad . \quad . \quad (2)$$

where $a, b, c$ are constants, then the equation (1) is

$$\frac{dv}{du}=\frac{C[\beta^2N^*+\beta\gamma P^*+\gamma^2Q^*]-A\gamma[\beta R^*+\gamma S^*]}{\gamma B[\beta R^*+\gamma S^*]-D[\beta^2N^*+\beta\gamma P^*+\gamma^2Q^*]} \quad . \quad . \quad . \quad (3)$$

where

$$A=b_1\gamma-c_1\beta \qquad C=a_1\gamma-c_1\alpha$$
$$B=b_2\gamma-c_2\beta \qquad D=a_2\gamma-c_2\alpha$$

and

$$N^*=N\left(\frac{\alpha}{\gamma}\right),\ P^*=P\left(\frac{\alpha}{\gamma}\right),\ Q^*=Q\left(\frac{\alpha}{\gamma}\right),\ R^*=R\left(\frac{\alpha}{\gamma}\right),\ S^*=S\left(\frac{\alpha}{\gamma}\right)$$

Transforming now to parallel axes, taking as the new origin o coordinates the point where the lines $\alpha=0$ and $\gamma=0$ meet, we find the new equation by substituting

$$u=\frac{(a_2c_3)}{(a_1c_2)}+u'\ ,\ v=\frac{(a_3c_1)}{(a_1c_2)}+v$$

$(a_2c_3)=a_2c_3-a_3c_2$, etc.

In this way, we get

$$\alpha=a_1u'+a_2v'\ ,\ \beta=b_1u'+b_2v'+\varrho=\beta'+\varrho\ ,\ \gamma=c_1u'+c_2v'$$

$\varrho$ being a constant, and

$$A = (b_1\, c_2)\, v' - c_2\, \varrho \qquad C = (a_1\, c_2)\, v'$$
$$B = -(b_1\, c_2)\, u' - c_2\, \varrho \quad D = -(a_1\, c_2)\, u'$$
$$N\left(\frac{\alpha}{\gamma}\right) = N\left(\frac{a_1\, u' + a_2\, v'}{c_1\, u' + c_2\, v'}\right) = N_0\,, \quad P\left(\frac{\alpha}{\gamma}\right) = P_0 \text{ etc.}$$

where $N_0\, P_0$ etc. are homogeneous functions of $u'$ and $v'$ of degree zero.

Hence, if we arrange according to the degrees of $u'$ and $v'$ the numerator takes the form

$$\begin{aligned}
&(a_1\, c_2)\, v'\, [N_0\, \beta'^2 + P_0\, \beta'\gamma + Q_0\, \gamma^2] \\
&- (b_1\, c_2)\, v_1\, [R_0\, \beta'\gamma + S_0\, \gamma^2] \\
&+ \varrho(a_1\, c_2)\, v'\, [2\, N_0\, \beta' + P_0\gamma] \\
&- \varrho(b_1\, c_2)\, R_0\, v'\gamma \\
&+ \varrho\, c_1\, [R_0\beta'\, \gamma + S_0\gamma^2] \\
&+ \varrho^2\, (a_1\, c_2)\, N_0\, v' \\
&+ \varrho^2\, c_1\, R_0\, \gamma
\end{aligned}$$

and in the same way the denominator may be written

$$\begin{aligned}
&(a_1\, c_2)\, u'\, [N_0\, \beta'^2 + P_0\, \beta'\, \gamma + Q_0\, \gamma^2] \\
&- (b_1\, c_2)\, u'\, [R_0\, \beta'\, \gamma + S_0\, \gamma^2] \\
&+ \varrho\, (a_1\, c_2)\, u'\, [2\, N_0\, \beta' + P_0\, \gamma] \\
&- \varrho\, (b_1\, c_2)\, R_0\, u'\, \gamma \\
&- \varrho\, c_2\, [R_0\, \beta'\gamma + S_0\, \gamma^2] \\
&+ \varrho^2\, (a_1\, c_2)\, N_0\, u' \\
&- \varrho^2\, c_2\, R_0\, \gamma.
\end{aligned}$$

If we examine these values it is evident that the equation (3) reduces to

$$\frac{dv'}{du'} = \frac{K_1 + M_2 + v'(N_2 + c)}{H_1 + L_2 + u'(N_2 + c)} \quad \ldots\ldots\ldots \quad (4)$$

where $c$ represents a constant, $H_1$ and $H_2$ homogeneous functions of the first degree and $L_2\, M_2\, N_2$ homogeneous functions of the second degree.

From the values

$$\begin{aligned}
H_1 &= -\varrho^2 c_2 R_0 \gamma \\
K_1 &= \quad \varrho^2 c_1 R_0 \gamma \\
c &= \quad \varrho^2 (a_1 c_2) N_0
\end{aligned}$$

we may readily induce that if, in (1) $R(x)$ is absent $H_1$ and $K_1$ must be zero and if in (1) $N(x)$ is absent, we have $c = 0$.

The preceding considerations furnish the inference that every homographic substitution applied to an equation (1), followed by a transformation to parallel axes through the point $\alpha = \gamma = 0$ gives necessarily an equation of the form (4).

3. Now we will show that where a differential equation of this form (4) is given, there always exists a homographic substitution by which this equation may be reduced to the form (1).

For let

$$u' = \frac{1}{y} \qquad v' = \frac{x}{y}$$

then we have

$$K_1 = K_1\,(u'v') = K_1\left(\frac{1}{y}\,,\,\frac{x}{y}\right) = \frac{1}{y}\,K_1\,(1\,,\,x)$$

$$M_2 = M_2\,(u'v') = M_2\left(\frac{1}{y}\,,\,\frac{x}{y}\right) = \frac{1}{y^2}\,M_2\,(1\,,\,x)$$

etc. Thus (4) reduces to

$$\frac{dy}{dx} = \frac{\{H_1(1,x)+c\}y^2+L_2(1,x)y+N_2(1,x)}{\{xH_1(1,x)-K_1(1,x)\}y+xL_2(1,x)-M_2(1,x)} \quad . \; . \; . \quad (5)$$

which is of the same form as the differential equation (1).

4. Therefore we have proved this:

*Theorem.* The necessary and sufficient condition that a differential equation of the first order and the first degree, having a singular point in the origin of coordinates, may be reduced by a homographic substitution to an equation (1) is that it may be written in the form

$$\frac{dy}{dx} = \frac{K_1+M_2+y(N_2+c)}{H_1+L_2+x(N_2+c)} \quad . \; . \; . \; . \; . \; . \; . \quad (6)$$

*Corollary* 1. The necessary and sufficient condition that a differential equation of the same kind may be reduced by a homographic substitution to an equation of Riccati is that it has the form

$$\frac{dy}{dx} = \frac{M_2+y(N_2+c)}{L_2+x(N_2+c)}. \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

*Corollary* 2. The necessary and sufficient condition that a differential equation of the same kind may be reducible by a homographic substitution to a linear equation is that it has the form

$$\frac{dy}{dx} = \frac{M_2+yN_2}{L_2+xN_2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

5. With respect to the equation (8) we may remark that it is equivalent with

$$\frac{dy}{dx} = \frac{M_1 + y\,N_1}{L_1 + x\,N_1}$$

as the numerator and the denominator of the second member may be divided by the same homogeneous function of the first degree. In the special case that $L_1 = a_1x + b_1y$, $M_1 = a_2x + b_2y$, $N_1 = c_1x + d_1y$ the tangents to the integral curves in the different points of the line $y = mx$, meet in the pole

$$X = -\frac{a_1 + b_1 m}{c_1 + d_1 m} \qquad Y = -\frac{a_2 + b_2 m}{c_1 + d_1 m}$$

Hence the locus of these poles for all the rays of the pencil $y = mx$ is the polar line

$$\begin{vmatrix} X & Y & 1 \\ a_1 & a_2 & -c_1 \\ b_1 & b_2 & -d_1 \end{vmatrix} = 0$$

This is the case in the examples II—VI given by Prof. DE VRIES. As to the examples I and VII we have respectively

$$L_1 = x \qquad M_1 = x + 2y \qquad N_1 = \frac{xy^2 + y^3}{x^2}$$

$$L_1 = 0 \qquad M_1 = y \qquad N_1 = \frac{x^2}{y}$$

**Physics.** — "*Contribution to the theory of binary mixtures.*" XIV. By Prof. J. D. VAN DER WAALS.

(DOUBLE RETROGRADE CONDENSATION).

Before proceeding to the discussion of the significance of negative value of $\varepsilon_1$ and $\varepsilon_2$, I shall make a few remarks to elucidate what was mentioned in the preceding contribution — and that chiefly on the shape of the surface of saturation in the cases represented by figs. 39 and 40, and the relative position of the three-phase-pressure with respect to the sections of that surface for given value of $x$.

In case of complete miscibility such a section of the surface of saturation consists of a vapour branch and a liquid branch, which have a continuous course, in which the pressure gradually increases with ascending $T$, and which for certain value of $T$, which may be indicated by $T_i$, pass into each other continuously. The pressure must then before have had a maximum on the liquid branch, and then decrease. It passes into the pressure of the vapour branch at $T_i$. This gradual merging of the two branches into each other continues to exist also for non-complete miscibility.

In the case of fig. 39 the upper sheet of the surface of saturation undergoes, however, first of all a modification, which, however, is