Citation:

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Astronomy. — "*On periodic orbits of the type Hestia.* By Dr. W. DE SITTER. (Communicated by Prof. J. C. KAPTEYN).

The problem, of which some particular solutions will be treated here, is the following. Two material points S and J, having the masses 1 and μ, move with uniform angular velocity $n' = 1$ in circles in one and the same plane round their centre of gravity. The constant distance SJ is adopted as unit of length. Another material point P, with an infinitely small mass, moves in the same plane under the influence of the Newtonian attractions of S and J. This is the problem which has (for $\mu = 0.1$) been so exhaustively treated by DARWIN in Vol. XXI of the Acta Mathematica. The particular solutions which are treated below are those in which the orbit of P is periodic and its limit for Lim. $\mu = 0$ is an ellipse with a small excentricity, described round S as a focus with a mean motion not differing much from 3. If this limiting orbit (i.e. the undisturbed orbit) is a circle, then the solution is, in POINCARÉ's phraseology, of the first sort (*sorte*), and its period is $T = \frac{2\pi}{n-1}$. If the excentricity of the undisturbed orbit differs from zero, the solution is of the second sort, and the limiting value of the period for Lim. $\mu = 0$ is Lim. $T' = 2\pi$. These solutions of the second sort are at the same time of the second genus (*genre*) relatively to those of the first sort.

The solutions of the first sort are the orbits of DARWIN's "Planet A". This family of orbits undergoes within the range here considered a transition from stability to instability, which has been discussed by POINCARÉ in an investigation contained in the articles 383 and 384 of his "Méthodes Nouvelles" (Vol. III, p. 355—361). The results there reached will be derived here by a different (and, as it seems to me, simpler) reasoning.

DARWIN's work also presents an example of an orbit of the second sort, viz. the orbit figured by him on page 281 and designated as $x_0 = -.337$. Although POINCARÉ proves the existence of solutions of this kind, he seems to have overlooked the fact that DARWIN had actually computed one of them.

These solutions and their stability I wish to consider from the point of view of the general theory developed by POINCARÉ in the first and third volumes of the "Méthodes Nouvelles". The following is a summary of those general theorems, proved by POINCARÉ, which will be used here. They are true for every problem capable of being reduced to two degrees of freedom, containing one variable parameter, and admitting for each value of this parameter a finite number of periodic solutions. It need hardly be mentioned that their valency is restricted to a certain domain of the several variable quantities of the problem, of which it will however not be necessary to transgress the limits.

A periodic solution is completely determined by the values of the parameter and of one constant of integration or "element". The periodic solutions occur in families, the members of which are classified according to increasing or decreasing values of the parameter. These families may be graphically represented by curves $\Phi(\varkappa, \beta) = 0$, where $\varkappa$ is the parameter of the problem and β the determining element.

The stability or instability is determined by a certain quantity α, which is by POINCARÉ called the characteristic exponent. If the period is T, then values of α differing by a multiple of $\frac{2\pi i}{T}$, must be considered as identical. The following three cases are possible:

αT purely imaginary . . . . . the solution is *stable*
αT real . . . . the solution is *evenly unstable*
αT complex, with imaginary part $= \pi i$: the solution is *unevenly unstable*[1]).

A solution having the period T can as well be conceived to have the period $T' = 2T$. If it is unevenly unstable with reference to the period T, it is evenly unstable with reference to the period T'.

Within each family the exponent α and the period T vary continuously with the parameter $\varkappa$. The product αT and the differential coefficient $\frac{d\Phi}{d\beta}$ become equal to zero for the same values of $\varkappa$. The curve $\Phi = 0$ then either has a multiple point, or is tangent to a line $\varkappa =$ const. The family splits into two branches, or, which comes to the same thing, two families have one member in common. If $(\varkappa_0 \beta_0)$ is the point representing this common member, then we have the following rules.

The number of branches of the curve $\Phi = 0$ (i. e. the number of families of periodic solutions) for $\varkappa > \varkappa_0$ differs by an even number from the number of branches for $\varkappa < \varkappa_0$.

The branches which part from the point $(\varkappa_0 \beta_0)$ towards the direction of increasing $\varkappa$ are alternately stable and evenly unstable[2]). The

[1]) The names *even* and *uneven instability* have been introduced by DARWIN. POINCARÉ distinguishes them as instability of the first and second "classe". The relation of DARWIN's quantity c to the exponent α is given by the formula $\alpha T = i\pi c$.

[2]) To avoid circumlocution I speak of "stable and unstable branches", meaning branches whose points represent stable and unstable solutions respectively.

same thing is true of the branches on which $\varkappa$ decreases. The two branches between which lies the part of the line $\varkappa = \varkappa_0$ on which $\beta < \beta_0$, are either both stable, or both unstable, and similarly the two branches enclosing the other half of the line $\varkappa = \varkappa_0$. If T is the period of one of the branches and T' of another, and if T_0 and T'_0 are the values of these periods in the point $(\varkappa_0\, \beta_0)$, then T_0 and T'_0 are mutually commensurable. If T''_0 is their least common multiple, then $\alpha_0 T''_0 = 0$. If e. g. $T'_0 = 2.T_0$, then the instability is even with reference to the period T'.

As an illustration of these general rules I may be allowed to mention a few of the simplest cases.

1. The curve $\Phi = 0$ is tangent to the line $\varkappa = \varkappa_0$. There are two families, springing from a common member, which come into existence at this value of the parameter. One of them is stable, and the other is evenly unstable. An example of this is presented by DARWIN's families B and C of satellites.

2. The curve has a double point. Two families are "crossing" each other, at the same time exchanging their stability.

3. The curve consists of one branch tangent to the line $\varkappa = \varkappa_0$ and another branch intersecting the first in the point of contact. The two families which come into existence at this value of the parameter are both stable or both unstable. The third family, which exists both for $\varkappa > \varkappa_0$ and for $\varkappa < \varkappa_0$, becomes stable if it was unstable and unstable if it was stable.

The cases 2 and 3 are the only ones occurring in the present investigation.

The proof of the above supposes that the problem can be reduced to the second order, so that there are only two characteristic exponents ($+ \alpha$ and $- \alpha$). The choice of the parameter is determined by the way in which this reduction is effected, or is conceived to be effected. DARWIN uses the integral of JACOBI for this reduction. Consequently his parameter is the constant C to which this integral is equal. This constant C is a function of the two elements a and e. The first of these can be replaced by the mean motion n, or by the period $T = \dfrac{2\pi}{n-1}$. In consequence of the reduction of the problem by means of the integral of JACOBI one of these elements, say T, is eliminated. This therefore appears no longer as an arbitrary constant of integration, but is entirely determined by C and e. On the other hand C is entirely determined by T and e. Now DARWIN's calculations show that T continually increases if C decreases. It is therefore irrevelant for our purpose whether we consider C or T as the

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parameter of the problem. The parameter which I will use here is $T' = 2\,T$. This change from C to T' can also be conceived as no more than a simplification of language. Instead of saying: "the solution corresponding to the value of C for which the period of the solution of the first sort is $^1/_2\,T'$", I say: "the solution corresponding to the value T'".

In DARWIN's work μ has the constant value 0.1. If now we choose a convenient element ξ, we can conceive the curves $\Phi\,(T', \xi)$ to be drawn. Next imagine the same thing to be done for other values of μ, and take μ, T' and ξ as rectangular coordinates. The curves $\Phi\,(T', \xi)$ belonging to the various values of μ then produce a surface, every point of which represents a periodic solution.

If, on the other hand, we take for T' a fixed value T'_1, considering μ as the variable parameter, then we have another problem, also admitting families of periodic solutions, which can be represented by curves $\psi\,(\mu, \xi) = 0$. If T'_1 varies these curves describe again the same surface. The form of this surface will now be investigated. Its section by the plane $\mu = 0.1$ then gives all periodic solutions of DARWIN's problem.

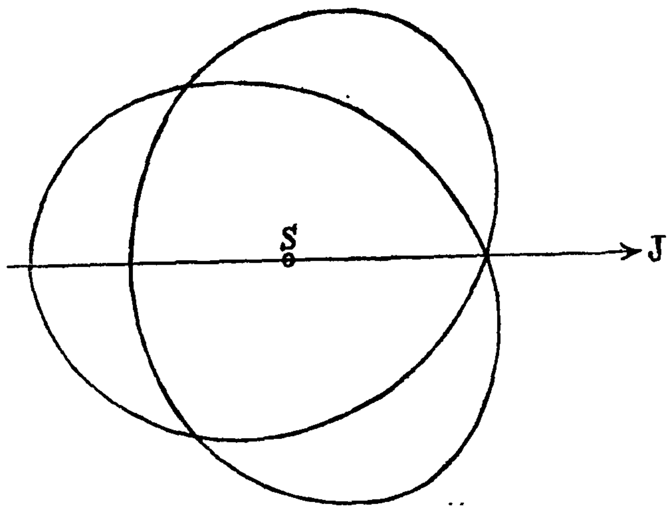
The element which I will use is $\xi = e_0 \cos \tilde{\omega}_0$, where e_0 is the excentricity and $\tilde{\omega}_0$ the longitude of the perihelion of the undisturbed orbit, which is the limit of the orbit of P for lim. $\mu = 0$. The longitude $\tilde{\omega}_0$ is counted from a *fixed* axis which at the beginning of the period co-incides with SJ. The orbit of P is not periodic unless $\tilde{\omega}_0$ has one of the two values 0 or π. Moreover at the beginning of the period P must be on the line SJ, i.e. there must be either opposition or conjunction.

Solutions of the first sort are characterised by $\xi = 0$. These solutions can have any period, therefore the whole plane $\xi = 0$ is a part of our surface. The line $\xi = 0$, $\mu = 0.1$ represents DARWIN's family A. For a value of $T' = 2T$, which lies between 330° and 354°, i.e. between $1.83\,\pi$ and $1.97\,\pi$, this family loses its stability and becomes unevenly unstable. So there must be another family which at this point has a member in common with the family A. This new family must have the period T', and is therefore of the second sort. If for the sake of argument we assume the change of stability to take place at the value $T' = 1.9\,\pi$, then we know of the branch of the curve $\Phi = 0$, which represents this family, that for $T' < 1.9\,\pi$ it is evenly unstable and for $T' > 1.9\,\pi$ it is stable.

Now there are only four possible periodic solutions of the second sort, distinguished by the following positions of P at the beginning of the period:

B : P in opposition in aphelion $(\tilde{\omega}_0 = 0,\ \xi = + e_0)$
B' : ,, ,, ,, ,, perihelion $(\tilde{\omega}_0 = \pi,\ \xi = - e_0)$
C : ,, ,, conjunction ,, perihelion $(\tilde{\omega}_0 = 0,\ \xi = + e_0)$
C' : ,, ,, ,, ,, aphelion $(\tilde{\omega}_0 = \pi,\ \xi = - e_0)$

With reference to rotating axes, of which the axis of x co-incides with SJ, the orbits B and B' are identical, and similarly C



Orbit of family B or B'

Fig. 1.

and C'. The orbits B and B' are of the form represented in fig. 1. The orbits C and C' are of the same form, rotated through 180°, i.e. with the double point away from J.

The families B and B' are stable, C and C' are unstable. This is easily found by considering the equation which determines the exponent α. This equation is (see POINCARÉ, Acta Math. XIII, p. 134):

$$n_1^2\,\alpha^2 = \frac{d^2\,\psi}{d\tilde{\omega}_2^{\,2}}\,(n_1^2\,C_{22} - 2\,n_1\,n_2\,C_{12} + n_2^2\,C_{11})$$

Now using the variables employed by POINCARÉ l. c. pages 128 and 171, we find easily

$$n_1 = -1 \qquad n_2 = 3 \qquad C_{11} = C_{12} = 0 \qquad C_{22} = -3\,x_2^{-4}.$$

If further in ψ (i. e. the average value of the perturbing function over one period) we neglect the terms which contain a higher power of e than the second, we find

$$\psi = \mu\,Ke^2\cos\varepsilon \qquad\qquad \varepsilon = \tilde{\omega}_2 + 3\,\tilde{\omega}_1$$

where ε is the mean longitude of P at the beginning of the period, and K is a positive constant.

We find thus

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$$a^2 = 3 \mu K e^2 x_2^{-4} \cos \varepsilon.$$

Thus, for positive values of μ, a^2 is negative, and therefore the orbit is stable, when there is opposition at the beginning of the period.

For positive values of μ therefore BB' is stable and CC' is unstable, for negative values [1]) of μ BB' is unstable and CC' is stable. It is evident that, for $\xi = 0$, B and B' co-incide, and similarly C and C'. The branch of $\Phi = 0$ which intersects $\xi = 0$ in the point $T' = 1.9\,\pi$ therefore represents either the family BB' or the family CC'. In the first case it is stable, and therefore it must on both sides of the point of intersection bend round towards the right. In the other case it is unstable and encloses the stable part of the line $\xi = 0$.

Now DARWIN has, for $C = 39.0$, i. e. $T' = 1.97\,\pi$, actually computed and drawn an orbit, which shows the form of fig. 1, viz.: the orbit $x_0 = -.337$ which has already been quoted. This orbit thus belongs to the family B, but it also belongs to B'. It belongs to B if P is in aphelion at the beginning of the period and in perihelion in the middle of the period (being at both times in opposition to J), and to B' in the opposite case. The branch of the curve $\Phi = 0$ which passes through the point $T' = 1.9\,\pi$ therefore represents the family BB', and not CC'. Consequently it is stable, and that part of the section of our surface by the plane $\mu = 0.1$, which lies to the left of the line $T' = 2\,\pi$, is thereby completely determined. This section is represented in Fig. 2. Stable families are there, and in the following figures, represented by heavy full lines, unevenly unstable families by broken lines, and evenly unstable ones by dotted lines.

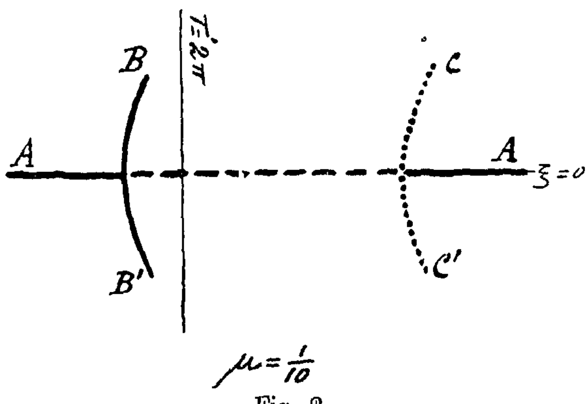

Fig. 2.

We next consider the section of our surface by the plane $\mu = 0$.

1) The meaning of a negative value of μ is that the force emanating from J is repulsive, the force from S remaining attractive.

We know then that there are stable periodic solutions of the first sort with an arbitrary period, and of the second sort with the period $T' = 2\pi$ and an arbitrary excentricity. The section therefore consists of the line $\xi = 0$ and the part of the line $T' = 2\pi$ between the points $\xi = +1$ and $\xi = -1$. I wish, however, to confine myself to *small* values of ξ. This section is represented in Fig. 3.

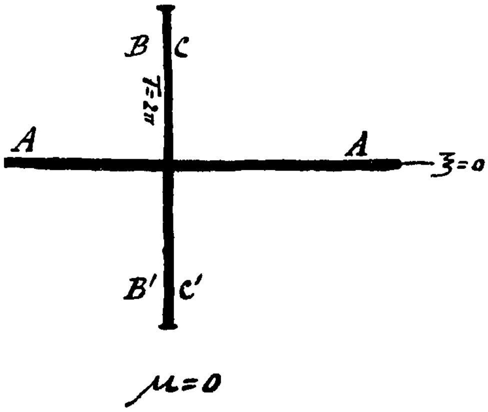

Fig. 3.

Next consider the section by a plane $T' = T_1'$, where $1.9\pi < T_1' < 2\pi$, and the curves $\psi(\mu, \xi) = 0$ in that plane. The line $\xi = 0$ is a part of this curve. The lower part of this line is stable, the upper part is unevenly unstable. In the point where the transition to instability takes place the line $\xi = 0$ is intersected by the branch of $\psi = 0$ representing the family BB'. This family being stable, that branch must on both sides of the point of intersection bend upwards, as is represented in fig. 4*a*.

Consider now the section of our surface by a plane parallel to, and at a very small distance from, $\xi = 0$. The orbits represented by the curves $\chi(\mu, T')$ in this plane are all of the second sort. We can imagine these orbits to arise by a variation of μ from the undisturbed periodic orbit of the second sort. They then appear as

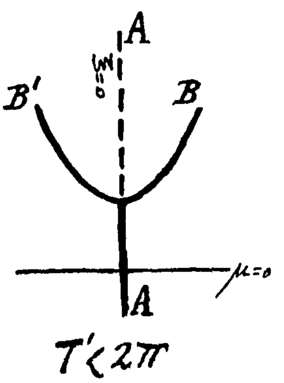

Fig. 4*a*.

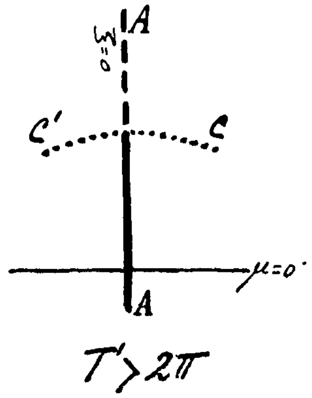

Fig. 4*b*.

solutions of a problem, in which the parameter is μ, ξ being kept constant, and thus T' (or C) now is our element. These solutions have been studied by SCHWARZSCHILD (Astr. Nachr. 3506). For $\mu = 0$ the period is 2π. For small values of μ there are (for each value of ξ) two solutions, viz. B and C when ξ is positive, B' and C'

when it is negative. The curve $\chi = 0$ thus consists of two branches, both passing through the point $\mu = 0$, $T' = 2\pi$, and there exchanging their stability. Since now it has already been shown that the stable branch B is, for positive values of μ, situated on the left side, the unstable branch C must be on the right side. The curves are represented in fig. 5.

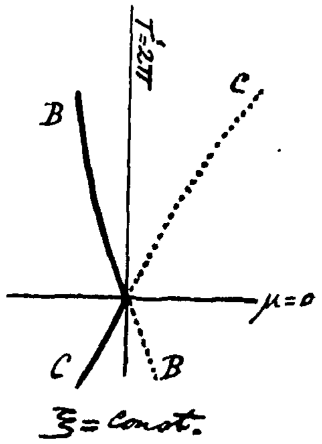

Fig. 5.

Our surface has thus been shown to consist of the plane $\xi = 0$ and of two sheets, which pass through the line $\mu = 0$, $T' = 2\pi$, and then deviate to the left and to the right of the plane $T' = 2\pi$. The points of the left-hand sheet represent the stable family BB', those of the right-hand sheet the unstable family CC'. This latter sheet therefore intersects the plane $\mu = 0.1$ in a curve which on both sides of its point of intersection with the line $\xi = 0$ bends off towards the right. In this same point of intersection the family A regains its stability, the stable part of the line $\xi = 0$, which represents this family, being enclosed between the two unstable branches of the section just considered. This state of things is rendered in the right-hand part of fig. 2. Also the form of the section of the surface by a plane $T' = T_2' > 2\pi$, will need no further explanation. It is represented in fig. 4*b*. Whether this right hand sheet does reach up to the plane $\mu = 0.1$, so as to produce a real section, cannot be decided by this reasoning. If there is a point of intersection with the line $\mu = 0.1$, $\xi = 0$, this must correspond to a value of T' exceeding $414.°3 = 2.23\,\pi$, since for this value the family A is still unevenly unstable, as is shown by Darwin's work. That the left-hand sheet does actually intersect the plane $\mu = 0.1$ is shown by the existence of Darwin's orbit $x_0 = -.337$, belonging to the family BB' (and also by the change of stability of the family A).

Thus all results have been derived which have been found by Poincaré in the "Méthodes Nouvelles", already quoted. Naturally Poincaré also must leave the question, whether his results still hold for $\mu = 0.1$, unanswered.

It is not uninteresting to consider the solutions B and C from the point of view of the theory of perturbations. This can, of course, not teach us anything about their stability, but it will give information about the form of the curves $\chi(\mu, T') = 0$ and $\psi(\mu, \xi) = 0$ for small values of μ and ξ. The period of the undisturbed solution is 2π, By the perturbing influence of J this is changed to $T' = 2\pi + \tau$. The conditions that the perturbed orbit shall be periodic are:

$$\int_0^{T'} \frac{d\tilde{\omega}}{dt}\, dt = \tau \qquad\qquad \int_0^{T'} \frac{d\lambda}{dt}\, dt = 6\,\pi + \tau,$$

where λ is the mean longitude of P. For the computation of the integrals we must use the mean motion affected by perturbations, i. e. $n = 3 + \sigma$. The left-hand members of these equations of condition are therefore functions of τ and σ, and these two unknowns can be determined from them.

If in these equations of condition we neglect the square and higher powers of e, they become

$$\left.\begin{aligned} \tau &= \frac{na}{4}(2\,\pi + \tau)\,\mu\,[B^{(1)} \pm \{21\,A^{(3)} + 10\,A_1^{(3)} + 2\,A_2^{(3)}\}] \\ 6\pi + \tau &= (3 + \sigma)\,(2\pi + \tau) - na\,(2\pi + \tau)\,\mu\,A_1^{(0)} \end{aligned}\right\} . \quad (1)$$

The upper sign in these equations must be used for the family $C\,C'$, the lower sign for $B\,B'$. The sum within the $\{\ \}$ being larger than $B^{(1)}$, we find that for the family BB' τ is negative, while for CC' it is positive, as has also been found above. Further the first equation shows that the numerical value of the differential coefficient $\frac{d\tau}{d\mu}$ for the first family (BB') decreases if μ increases, while for the other family it increases. Thus the left-hand branch of $\chi\,(\mu,\, T') = 0$ has its concave side towards the line $T' = 2\pi$, and the right-hand branch its convex side, as is shown in fig. 5.

In the numerical computation we must not forget that the formulas (1) can only be considered as approximatively true. The solution of the equations is easily effected by means of the tables of Runkle, the argument for the determination of the different functions $A_p^{(i)}$ being computed by

$$n = 3 + \sigma \qquad n^2\, a^3 = f = \frac{10}{11}.$$

I find in this manner for the two families:

$$B:\quad \frac{\tau}{2\pi} = -\,0.085 \qquad T' = 1.83\,\pi$$

$$C:\quad \frac{\tau}{2\pi} = +\,0.29 \qquad T' = 2.58\,\pi$$

These are the periods of those orbits of the two families, which have $\xi = 0$, and which therefore co-incide with a member of the family A, whose period is $T = \frac{1}{2} T'$. DARWIN's computations show that the value of T' for which the families A and B co-incide must lie between 1.836 π and 1,97 π. The point of co-incidence of A and C is outside the region explored by DARWIN, the corresponding value of T' must therefore be larger than 2.23 π.

If in the equations (1) we take account of the square of e, the right-hand member of the first must be multiplied by $\sqrt{1-e^2}$. In the second $A_1^{(0)}$ must be replaced by

$$A_1^{(0)} + \frac{1}{4} e^2 \left(B_1^{(1)} \pm \{31\, A_1^{(3)} + 24\, A_2^{(3)} + 6\, A_3^{(3)}\} \right)$$

and $^1/_2 e^2 \tau$ must be added to the second member. Now if we take $T' =$ const. then τ is constant and also σ can be taken to be constant. The second equation (1) then is of the form

$$\text{const.} = \mu\,(P + Qe^2) \; . \; . \; . \; . \; . \; . \; . \; . \qquad (2)$$

Now we have $\xi^2 = e^2$, therefore (2) is approximately the equation $\psi(\mu, \xi) = 0$. For the family BB' P and Q are of opposite signs, for CC' they have the same sign. Thus the form of these curves as drawn in the figures $4a$ and $4b$ is confirmed. [1])

Physics. — *"Contribution to the theory of binary mixtures.* IV." By Prof. J. D. VAN DER WAALS.

Continued, see p. 849 vol. IX.

THE BINODAL CURVE.

We might think that for the determination of the binodal curve we could follow the following course. It is required for coexistence that besides the temperature three other quantities are equal, i. e. p, q and $M_1 \mu_1$. If we now also trace the lines on which $M_1 \mu_1$, is equal, we should have to seek in order to find a point of a binodal curve, the points satisfying the condition that the p, q and $M_1 \mu_1$ lines passing through this point intersect in still another point of the field. This search, however, being exceedingly difficult would give moreover no clear survey of the results. We shall, therefore not follow this course. Still I shall make some prefatory remarks on the course of this third group of lines. For it is by no means devoid of interest to know in which phases of a binary system the

[1]) This last paragraph has been added in the English translation.