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ratio of the amplitudes of the first or second sound to that of the third $a = 7$, and the ratio of the frequencies $b = 2$, the ratio of the intensities is $a^2 b^2 = 196$. Hence the third sound is at its maximum still about 200 times feebler than the first or second.

While the above given figures refer to the objective intensities, a comparison of the intensities of perception is still much less in favour of the third sound, since a tone of frequency 50 per second has objectively to be a little over a hundred times stronger [1]) than a tone of 100 vibrations a second, in order to produce an equally strong auditory impression. Consequently, if the third sound attains such an intensity that it is just audible still, the first and second sounds may be 20.000 times weakened, before also the auditory impression they produce, vanishes.

This explains the difficulty of the investigation by the method of auscultation. GIBSON [2]) emphasises this particularly and says that in order to hear the sound, accidental sounds must be excluded as much as possible, while one has to strain one's attention during the interval in which the sound occurs. Although the cardiophonograms leave no doubt as to the existence of the third heart sound with W i, we have been unable to hear it by means of a stethoscope.

Regarding the explanation of the third sound we refer to the above mentioned more extensive paper which will shortly be published elsewhere. Here we will only state our conclusion that the sound cannot be put on a line with a prae-systolic murmur of the mitral valve, nor with a duplication of the second sound by non-simultaneous action of the aortal and pulmonal valves, but that it is probably caused by a second vibration of the valvulae semilunares aortae and must be regarded as a phenomenon of pretty common occurrence.

Astronomy. — "*On some points in the theory of Jupiter's satellites.*" By Dr. W. DE SITTER. (Communicated by Dr. E. F. VAN DE SANDE BAKHUYZEN).

The following pages contain a short account of some investigations, which will soon be published, together with other results, in N°. 17 of the publications of the astronomical laboratory at Groningen.

A few words are necessary in explanation of the notations em-

1) Calculated according to MAX WIEN, PFLÜGER's Arch. f. d. gesammte Physiol. Bd. 97. p. 1. 1903. H. ZWAARDEMAKER and F. H. QUIX give in ENGELMANN's Arch. f. Physiol. p. 25. 1904, differences in the same sense, but of a different order of magnitude.

2) l. c.

ployed. The notations used by different writers on the theory of the satellites are discordant in a most regrettable manner. The tables, both those of DAMOISEAU and of DELAMBRE, distinguish the four satellites by the numbers 1, 2, 3, 4. This example is followed by MARTH, and I have also in all my previous work on the satellites used this notation, as is also done by Mr. COOKSON in the discussion of his observations. The theoretical writers, on the other hand, LAPLACE, TISSERAND, SOUILLART use the suffixes 0, 1, 2, 3 or a corresponding number of accents. Another fundamental difference is in the designation of the perijoves. The letter ϖ in the writings of DAMOISEAU, MARTH, COOKSON and myself represents the "own" perijove; SOUILLART and TISSERAND use it for the osculating perijove. There are many more differences of this kind, which need not be enumerated here. Though thoroughly convinced of the great importance of a consistent notation, I am, reluctantly, compelled in this communication to depart from the notations employed by me elsewhere. In the first article of the present communication, which treats of a theoretical point, I have, to avoid the writing out at length of many well known formulas and results, closely followed TISSERAND's very clear argument in the fourth volume of his *Traité de Mécanique Céleste*. Accordingly *in this first article,* I will adopt TISSERAND's notation, with one exception. In the further articles I will return to the notation employed in my previous work.

1. *Theory of the libration.* As has been explained, the notations employed are TISSERAND's excepting the mean longitudes, which I denote by l_1, l_2, l_3 instead of by l, l', l''. In addition to the quantities F, F' G, G' defined by (19) page 11 [1]) I wish to introduce

$$G_1 = \frac{4a}{a'} - 3a' A^{(1)} - a\,a' \frac{\partial A^{(1)}}{\partial a}$$

$$G_1' = \frac{4a'}{a''} - 3a'' A'^{(1)} - a'a'' \frac{\partial A'^{(1)}}{\partial a'}.$$

TISSERAND assumes $G_1 = G$ and $G_1' = G'$, which is only approximately true. If it is not desired to introduce this approximation, then on page 11, formula (20) we must in R_1 replace G by G_1 and similarly in R_1' G' by G_1'.

The only further difference from TISSERAND's notation is in the definition of the libration. I put

1) The references of pages and formulas are to those of TISSERAND, volume IV.

$$\vartheta = l_1 - 3l_2 + 2l_3 + 180°, \quad [1]$$

TISSERAND, however, has

$$\vartheta = l - 3l' + 2l''.$$

The angle ϑ, as defined by [1] is the angle to which the name *libration* was first applied by LAPLACE, and which is by him called $\tilde{\omega}$. (*Mécanique Céleste,* Livre VIII, art. 15, *Oeuvres,* tome IV pages 75 and 79 of the edition of 1845).

The differential equation determining the libration is

$$\frac{d^2\vartheta}{dt^2} = -\beta^2 \sin\vartheta \quad [2]$$

This equation is derived by the combination of the three equations

$$\left.\begin{aligned} \frac{d^2l_1}{dt^2} &= -Q_1 \sin\vartheta \\ \frac{d^2l_2}{dt^2} &= -Q_2 \sin\vartheta \\ \frac{d^2l_3}{dt^2} &= -Q_3 \sin\vartheta \end{aligned}\right\} \quad [3]$$

We have thus

$$\beta^2 = Q_1 - 3Q_2 + 2Q_3 \quad [4]$$

From these equations the whole theory of the libration is derived in the well known manner, on which, however, I will not dwell, my sole object being at present the determination of the quantities Q_1, Q_2 and Q_3.

For that purpose we start from the formulas given by TISSERAND at the top of page 20, which must however be completed as follows:

$$\frac{d^2\rho}{dt^2} = -\frac{3}{a^2}\left(\frac{\partial R_1}{\partial \varepsilon} + \frac{\partial R_4}{\partial \varepsilon}\right). \quad [5]$$

and two similar equations for ρ' and ρ''.

Introducing the same auxiliary angles u and u' that are used by TISSERAND (formula (12) page 20), we get instead of TISSERAND's equations (B):

$$
\begin{aligned}
\frac{d^2\varrho}{dt^2} = {} & \frac{3}{2} m' n'^2 \left[F\,(k \sin u - h \cos u) + \frac{a}{a'} G_1\,(k' \sin u - h' \cos u) \right] \\
& - 3n \Big[a_{0,1} \left(\{k^2 - h^2\} \sin 2u - 2kh \cos 2u \right) \\
& \quad + \frac{m' \sqrt{a'}}{m \sqrt{a}} a_{1,0} \left(\{k'^2 - h'^2\} \sin 2u - 2h'k' \cos 2u \right) \\
& \quad - 2b_{0,1} \left(\{kk' - hh'\} \sin 2u - \{kh' + hk'\} \cos 2u \right) \Big]. \\
\frac{d^2\varrho'}{dt^2} = {} & - 3mn'^2 \left[G\,(k' \sin u - h' \cos u) + \frac{a'}{a} F\,(k \sin u - h \cos u) \right] \\
& + \frac{3}{2} m'' n'^2 \left[F'\,(k' \sin u' - h' \cos u') + \frac{a'}{a''} G'_1\,(k'' \sin u' - h'' \cos u') \right] \\
& + 6n' \Big[a_{1,0} \left(\{k'^2 - h'^2\} \sin 2u - 2k'h' \cos 2u \right) \\
& \quad + \frac{m \sqrt{a}}{m' \sqrt{a'}} a_{0,1} \left(\{k^2 - h^2\} \sin 2u - 2kh \cos 2u \right) \\
& \quad - 2b_{1,0} \left(\{kk' - hh'\} \sin 2u - \{kh' + hk'\} \cos 2u \right) \Big] \\
& - 3n' \Big[a_{1,2} \left(\{k'^2 - h'^2\} \sin 2u' - 2k'h' \cos 2u' \right) \\
& \quad + \frac{m'' \sqrt{a''}}{m' \sqrt{a'}} a_{2,1} \left(\{k''^2 - h''^2\} \sin 2u' - 2k''h'' \cos 2u' \right) \\
& \quad - 2b_{1,2} \left(\{k'k'' - h'h''\} \sin 2u' - \{k'h'' + h'k''\} \cos 2u' \right) \Big]. \\
\frac{d^2\varrho''}{dt^2} = {} & - 3m'n''^2 \left[G'\,(k'' \sin u' - h'' \cos u') + \frac{a''}{a'} F'\,(k' \sin u' - h' \cos u') \right] \\
& + 6n'' \Big[a_{2,1} \left(\{k''^2 - h''^2\} \sin 2u' - 2k''h'' \cos 2u' \right) \\
& \quad + \frac{m' \sqrt{a'}}{m'' \sqrt{a''}} a_{1,2} \left(\{k'^2 - h'^2\} \sin 2u' - 2k'h' \cos 2u' \right) \\
& \quad - 2b_{2,1} \left(\{k'k'' - h'h''\} \sin 2u' - \{k'h'' + h'k''\} \cos 2u' \right) \Big].
\end{aligned}
\qquad [6]
$$

To derive from these the formulas [3] we must for $h, k, h' \ldots$ substitute the values

$$
\begin{aligned}
h &= B \sin u + B_1 \sin u' \\
k &= B \cos u + B_1 \cos u',
\end{aligned}
\qquad \text{etc.} \quad . \; . \; . \; . \; . \; . \; . \quad [7]
$$

which are given by TISSERAND at the bottom of page 21. In the result we then reject all terms which do not contain the argument

$$u' - u = \vartheta + 180^\circ,$$

or its multiples. We thus find easily

$$\left.\begin{aligned}
\frac{d^2\varrho}{dt^2} &= \frac{3}{2} m'n^2 \left[FB_1 + \frac{a}{a'} G_1 B'_1 \right] \sin (u-u') \\
&\quad - 3n \left[a_{0,1} B_1^{\,2} + \frac{m'\sqrt{a'}}{m\sqrt{a}} a_{1,0} B_1'^{\,2} - 2b_{0,1} B_1 B_1' \right] \sin 2 (u-u') \\
&\quad - 6n \left[a_{0,1} BB_1 + \frac{m'\sqrt{a'}}{m\sqrt{a}} a_{1,0} B'B_1' - b_{0,1} (BB_1' + B_1 B') \right] \sin (u-u') \\
\frac{d^2\varrho'}{dt^2} &= - 3mn'^2 \left[GB_1' + \frac{a'}{a} FB_1 \right] \sin (u-u') \\
&\quad + \frac{3}{2} m''n'^2 \left[F'B' + \frac{a'}{a''} G_1' B'' \right] \sin (u'-u) \\
&\quad + 6n' \left[a_{1,0} B_1'^{\,2} + \frac{m\sqrt{a}}{m'\sqrt{a'}} a_{0,1} B_1^{\,2} - 2b_{1,0} B_1 B_1' \right] \sin 2 (u-u') \\
&\quad + 12n' \left[a_{1,0} B'B_1' + \frac{m\sqrt{a}}{m'\sqrt{a'}} a_{0,1} BB_1 - b_{1,0} (BB_1' + B_1 B') \right] \\
&\qquad \sin (u-u') \\
&\quad - 3n' \left[a_{1,2} B'^2 + \frac{m''\sqrt{a''}}{m'\sqrt{a'}} a_{2,1} B''^2 - 2b_{1,2} B'B'' \right] \sin 2 (u'-u) \\
&\quad - 6n' \left[a_{1,2} B'B_1' + \frac{m''\sqrt{a''}}{m'\sqrt{a'}} a_{2,1} B''B_1'' - b_{1,2} (B'B_1'' + B_1'B'') \right] \\
&\qquad \sin (u'-u) \\
\frac{d^2\varrho''}{dt^2} &= - 3m'n''^2 \left[G'B'' + \frac{a''}{a'} F'B' \right] \sin (u'-u) \\
&\quad + 6n'' \left[a_{2,1} B''^2 + \frac{m'\sqrt{a'}}{m''\sqrt{a''}} a_{1,2} B'^2 - 2b_{2,1} B'B'' \right] \sin 2 (u'-u) \\
&\quad + 12n'' \left[a_{2,1} B''B_1'' + \frac{m'\sqrt{a'}}{m''\sqrt{a''}} a_{1,2} B'B_1' - b_{2,1} (B'B_1'' + B_1'B'') \right] \\
&\qquad \sin (u'-u)
\end{aligned}\right\} \quad [8]$$

We now put

$$\sin (u-u') = \sin \vartheta$$
$$\sin 2 (u-u') = - 2 \sin \vartheta,$$

Further we introduce the approximate values of $B, B' \ldots$ which TISSERAND gives in the middle of page 22, viz.:

$$B' = m\, CG \qquad B_1' = m''\, CF' \qquad B_1 = B'' = 0, \ldots \quad [9]$$

where C is a constant, the value of which is indifferent to our argument, and can easily be derived by comparison with TISSERAND.

We then neglect the squares and products of B, B' ..., and also the difference of G_1 and G, and we put

$$n^2 a^3 = n'^2 a'^3 = n''^2 a''^3 = f, \quad . \; . \; . \; . \; . \quad [10]$$

which also is only approximately true, and

$$-\frac{3}{2}\frac{f}{a'} CF'G = K,$$

Introducing all these simplifications we find the equations (22) of TISSERAND, viz.:

$$\frac{d^2 l_1}{dt^2} = -\frac{m'm''}{a^2} K \sin \vartheta$$

$$\frac{d^2 l_2}{dt^2} = 3\frac{mm''}{a'^2} K \sin \vartheta$$

$$\frac{d^2 l_3}{dt^2} = -2\frac{mm'}{a''^2} K \sin \vartheta,$$

In comparing these with TISSERAND it must not be forgotten that our ϑ differs 180° from TISSERAND's. We have thus, if all the above mentioned approximations are introduced

$$Q_1 = \frac{m'm''}{a^2} K, \qquad Q_2 = -3\frac{mm''}{a'^2} K, \qquad Q_3 = 2\frac{mm'}{a''^2} K. \quad . \quad [11]$$

The values [9], however, are only approximately true; they contain only the perturbations of the first order in the masses. Nevertheless the deviations of the values of Q_i from the truth caused by the adoption of these approximate values, and similarly by [10] and by the neglect of difference of G and G_1, are not of a serious nature. The neglect of the terms of the second degree in B, B'... on the other hand, is very serious.

Now discarding all these simplifications, with the exception of $B_1 = B'' = 0$, which we continue to adopt, we find for the complete values of Q_1, Q_2, Q_3:

$$\left.\begin{aligned}
Q_1 &= -\frac{3}{2} m'n^2 \frac{a}{a'} G_1 B_1' - 6n\left[\frac{m'\sqrt{a'}}{m\sqrt{a}} a_{1,0}(B_1'^2 - B'B_1') + b_{0,1} BB_1'\right] \\
Q_2 &= +3\,mn'^2\, GB_1' + \frac{3}{2} m''n'^2\, F'B' + \\
&\quad + 12n'\,[a_{1,0}(B_1'^2 - B'B_1') + b_{1,0} BB_1'] + \\
&\quad + 6n'\,[a_{1,2}(B'^2 - B'B_1') + b_{1,2}\, B'B_1''] \\
Q_3 &= -3m'n''^2 \frac{a''}{a'} F'B' - 12n''\left[\frac{m'\sqrt{a'}}{m''\sqrt{a''}} a_{1,2}(B'^2 - B'B_1') + b_{2,1} B'B_1''\right].
\end{aligned}\right\} [12]$$

Using the numerical data adopted by SOUILLART, and putting

$$\mathrm{m}_1 = 10000\, m, \qquad \mathrm{m}_2 = 10000\, m', \qquad \mathrm{m}_3 = 10000\, m''.$$

we find from formula [11]

$$Q_1 = + 0.03201\, m_2\, m_3$$
$$Q_2 = - 0.03794\, m_1\, m_3$$
$$Q_3 = + 0.00994\, m_1\, m_2,$$

From the formulas [12], on the other hand, we have:

$$Q_1 = \{+ 0\;03009 - \cdot 00460\, m_1 - \cdot 01156\, m_2 - \cdot 00958\, m_3\}\, m_2\, m_3 =$$
$$= + 0\cdot 01815\, m_2\, m_3$$
$$Q_2 = \{- 0\cdot 03436 + \cdot 00389\, m_1 + \cdot 00933\, m_2 + \cdot 00809\, m_3\}\, m_1\, m_3 =$$
$$= - 0\cdot 02438\, m_1\, m_3$$
$$Q_3 = \{+ 0\cdot 00794 - \cdot 00020\, m_1 - \cdot 00016\, m_2 - \cdot 00042\, m_3\}\, m_1\, m_2 =$$
$$= + 0\cdot 00751\, m_1\, m_2.$$

The numerical coefficients depend almost exclusively on the ratios of the major axes, i.e. on the mean motions, and they can be taken as correct to the last figure given.

The corresponding periods, computed by the formula

$$T = \frac{2\pi}{\beta},$$

are, expressed in years:

from formula [11] . . . . $T = 6\cdot 318$

from formula [12] . . . . $T = 7\cdot 985$,

The difference is considerable.

The question naturally arises: why have these important terms of the second degree been overlooked by LAPLACE and SOUILLART? For LAPLACE, the answer is very simple: he has neglected the part R_4 of the perturbing function throughout. For SOUILLART it is different. It is one of SOUILLART's great merits to have discovered the importance of this same part of the perturbing function, especially for the determination of the quantities B, B'... The corrections which have been added by SOUILLART on this account to these coefficients, amount to a considerable part of the whole. Also SOUILLART evidently intended to find the expression for the period of the libration as completely as possible. On the pages 46 and 47 (Memoirs of the Royal Astronomical Society, Vol. XLV) he considers the different parts of the perturbing function, which can in the differential coefficients of the mean longitudes introduce the argument $l_1 - 3l_2 + 2l_3$. He, however, rejects them all, as giving negligible coefficients, and retains only the terms which had already been discovered by LAPLACE. Among the rejected terms are also the new terms treated above, which are discarded by SOUILLART on the ground that they are of the second degree in the excentricities (page 47, bottom). He here overlooks that in these terms, for the same reason as in those of the first degree, the excentricities must be replaced by their perturbations with the arguments u and u', in order to find the terms determining the libration. These terms thus are of the second degree, not in the excentricities, but in the quantities B, B'... and of *these* the squares are not negligible, as we have seen.

The question further arises: do not the terms of the third degree in the excentricities, i. e. those of the types

$$P e^3 \cos(2l' - l - \tilde{\omega}), \qquad Q e^2 e' \cos(2l' - l - 2\tilde{\omega} + \tilde{\omega}'),$$
$$R e^3 \cos(6l' - 3l - 3\tilde{\omega}), \qquad S e^3 \cos(4l'' - l - 3\tilde{\omega}), \text{ etc.}$$

also contribute appreciably towards the coefficients Q_i? To find the answer to this question I have computed all the terms of this kind in Q_1. These terms of the third degree, which are of the fourth order in the masses, are:

$$\delta Q_1 = \{+ \cdot 00012\, m_1^2 + \cdot 00079\, m_2^2 + 00034\, m_3^2 + \cdot 00061\, m_1 m_2 + + \cdot 00050\, m_1 m_3 + .00124\, m_1 m_3\}\, m_2 m_3 = + .00071\, m_2 m_3.$$

They are thus not wholly negligible. I have, however, not carried out the computation — which is rather complicated — for Q_2 and Q_3, nor have I computed the terms of the fourth degree (i. e. of the fifth order in the masses). The development of the period T in powers of the masses evidently converges very slowly, and the period computed by the formulas [12] may very well be erroneous by a few tenths of a year.

2. *The equations of the centre.* The large inequalities, which in the integration by the method of variation of elements appear as perturbations of the excentricities and perijoves (formula [7] above), are in practice added to the longitudes and radii-vectores, and the excentricities and perijoves are conceived to be affected by their secular, but *not* by their periodic perturbations. I now return to the notations used in all my other work on the satellites, and I denote the excentricities and perijoves, defined in this way, by E_i and Ω_i. We have then [1])

$$\left.\begin{array}{l} h_i = 2 E_i \sin \Omega_i = 2 \Sigma_j\, \tau_{ij}\, e_j \sin \tilde{\omega}_j \\ k_i = 2 E_i \cos \Omega_i = 2 \Sigma_j\, \tau_{ij}\, e_j \cos \tilde{\omega}_j \end{array}\right\} \quad \ldots\ldots [13]$$

The sums extend over the values of j from 1 to 4; e_i and $\tilde{\omega}_i$ are the "own" excentricities and perijoves of LAPLACE, the values of e_i are constant and $\tilde{\omega}_i$ are linear functions of the time. Further

[1]) These h_i and k_i are thus *not* the same quantities as those denoted by h, k, h'... by TISSERAND.

$\tau_{ii} = 1$, the other ratios τ_{ij}, and the motions $\frac{d\tilde{\omega}_i}{dt}$ depending on the masses. Thus if certain values of the masses are adopted, the ratios τ_{ij} are thereby determined. If then h_i and k_i of the four satellites are known from the observations, then from the eight linear equations [13] (consisting of two sets of four each, with the same coefficients) we can determine the eight unknowns $e_i \sin \tilde{\omega}_i$ and $e_i \cos \tilde{\omega}_i$, and from these again e_i and $\tilde{\omega}_i$. The method is exactly the same as the one used by me for the determination of the inclinations and nodes (see these Proceedings, 1906 March, pages 767—780). The values of h_i and k_i have been determined from the heliometer-observations made at the Cape Observatory, in 1891 by Sir DAVID GILL, and in 1901 and 1902 by Mr. BRYAN COOKSON. The results from these observations have been treated by the method just delineated, in two different suppositions regarding the masses, i. e. regarding the ratios

Satellite	Epoch	e			$\tilde{\omega}$			$\tilde{\omega}_{1900 \cdot 0}$	
		System I	System II	*p.e.*	System I	System II	*p.e.*	System I	System II
I	1891·75	0°036	0°036	±°009	158°	157°	± 15°	248°	235°
	1901·61	·055	·055	± 22	136	136	± 36	48	50
	1902·60	·022	·021	± 17	262	270	± 27	120	131
II	1891·75	0 018	0 020	±·006	169	166	± 16.	300	274
	1901·61	·020	019	± 14	318	·315	± 37	292	294
	1902·60	·026	026	± 9	302	301	± 24	261	267
	Mean	0·021	0·022					284	278
III	1891·75	0·086	0·086	±·003	179·7	179·6	± 2 0	201·4	200·4
	1901·61	·100	·101	± 9	198·2	198·1	± 5·6	193·9	193·8
	1902·60	·080	·080	± 6	219·0	218 8	± 4·0	212 2	212·3
	Mean	0·089	0·089					202·5	202·2
IV	1891·75	0·4284	0·4280	±·0015	142·28	142 29	± 0·20	148·19	147·83
	1901·61	·4228	·4216	± 30	148 92	149·05	± ·40	147·76	147·96
	1902·60	·4261	·4262	± 25	149·06	149·03	± ·34	147·20	147·28
	Mean	0·4258	0·4253					147.72	147·69

τ_{ij} and the motions $\frac{d\tilde{\omega}_i}{dt}$. The results are collected in the following table. The values of $\tilde{\omega}_i$ for 1900.0, given in the last two columns, have been derived from those for the individual epochs for each system separately by means of the motions $\frac{d\tilde{\omega}_i}{dt}$ corresponding to the assumed masses. The perijoves are counted from the assumed vernal equinox of Jupiter, whose longitude in 1900.0 is 135°.45.

The values of these elements, on which SOUILLART's theory is based, are:

	(1900·0)
$e_1 = 0\overset{\circ}{\cdot}001$	$\tilde{\omega}_1 = 305^\circ$
$e_2 = 0{\cdot}006$	$\tilde{\omega}_2 = 177$
$e_3 = 0{\cdot}064$	$\tilde{\omega}_3 = 206{\cdot}1$
$e_4 = 0{\cdot}4160$	$\tilde{\omega}_4 = 152{\cdot}69$

The results from the two systems are practically identical. The corrections to SOUILLART's values for the satellites II, III and IV, are considerable, and on the whole much larger than the deviations of the three epochs *inter se.* These corrections are thus undoubtedly real. The most remarkable of them is certainly the large own excentricity of II. The value of this element, assumed by DELAMBRE and DAMOISEAU is zero. The value used by SOUILLART in his theory is a pure arithmetical result, and has no weight whatever as a determination of the element. DAMOISEAU, however, has suspected the existence of an excentricity of practically the same amount as is found here. This is shown by the following quotation from his unpublished memoir, written in explanation of the construction of his tables, which I quote after SOUILLART [1]). DAMOISEAU says there: "Nous avons des motifs de soupçonner dans l'orbite du second satellite une équation du centre propre de 32^s en temps synodique, (ce qui correspondrait à une excentricité propre de 0.00032738), mais notre incertitude sur la position du périjove, dont le mouvement est encore à calculer par la théorie, nous a fait remettre cette recherche à un autre temps." This excentricity, expressed in arc is 0°.0188, and it is therefore practically the same as the value found by me. The reason adduced by DAMOISEAU for not using it in his tables sounds somewhat strange: as a matter of fact the motion of the perijove had been determined long ago by LAPLACE.

With regard to Satellite I it is clear that the apparent equations

[1]) *Mémoires des Savants étrangers,* tome XXX, page 28.

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$e_3 = 0\cdot064$	$\tilde{\omega}_3 = 206\cdot1$
$e_4 = 0\cdot4160$	$\tilde{\omega}_4 = 152\cdot69$

The results from the two systems are practically identical. The corrections to SOUILLART's values for the satellites II, III and IV, are considerable, and on the whole much larger than the deviations of the three epochs *inter se.* These corrections are thus undoubtedly real. The most remarkable of them is certainly the large own excentricity of II. The value of this element, assumed by DELAMBRE and DAMOISEAU is zero. The value used by SOUILLART in his theory is a pure arithmetical result, and has no weight whatever as a determination of the element. DAMOISEAU, however, has suspected the existence of an excentricity of practically the same amount as is found here. This is shown by the following quotation from his unpublished memoir, written in explanation of the construction of his tables, which I quote after SOUILLART [1]). DAMOISEAU says there: "Nous avons des motifs de soupçonner dans l'orbite du second satellite une équation du centre propre de 32^s en temps synodique (ce qui correspondrait à une excentricité propre de 0.00032738), mais notre incertitude sur la position du périjove, dont le mouvement est encore à calculer par la théorie, nous a fait remettre cette recherche à un autre temps." This excentricity, expressed in arc is 0°.0188, and it is therefore practically the same as the value found by me. The reason adduced by DAMOISEAU for not using it in his tables sounds somewhat strange: as a matter of fact the motion of the perijove had been determined long ago by LAPLACE.

With regard to Satellite I it is clear that the apparent equations

[1]) *Mémoires des Savants étrangers,* tome XXX, page 28.

of the centre derived from the observations — which moreover are only little larger than their probable errors — do not represent a true excentricity. It is not impossible that they are produced by the existence of surface markings on the disc of the satellite, causing the centre of light, which is observed by the heliometer, to be displaced relatively to the centre of gravity, the displacement being different at different epochs. Any attempt to explain the observed h_i and k_i on this hypothesis would, however, necessarily involve so many undeterminate quantities, that its success would be no proof of its representing a true fact of nature.

3. *Determination of the libration from the observations.*

In a communication made by me in 1905 to the "Nederlandsch Natuur- en Geneeskundig Congres", [1]) I have shown:

that the libration probably has an appreciable coefficient,

that the determination from the observations, not only of the phase and amplitude, but also of the period of the libration, is of the highest importance for the derivation of the masses, especially of the mass of Satellite I,

that this determination is possible from the observations made at the observatories at the Cape, Helsingfors and Pulkowa,

that most probably the period differs considerably from the value adopted by LAPLACE and SOUILLART, and

that this determination is intricately connected with an investigation of the long-periodic inequalities in the longitudes of the satellites, and that consequently the whole problem can only be solved by successive approximations.

In number 17 of the Publications of the Astronomical Laboratory at Groningen, which will soon be published, all these conclusions are confirmed and the successive approximations are carried out. In this communication I cannot dwell upon the details of this investigation, nor upon the difficulties which were encountered. I must confine myself to a brief statement of the results.

The observations used are the heliometer-observations of the Cape Observatory already quoted above, and further photographic plates taken at Helsingfors in the years 1892—93, 1893—94, 1894—95, 1895—96 and 1897, at Pulkowa in 1895—96, 1897 and 1898, and at the Cape in 1904. I thus had at my disposition ten oppositions

[1]) *"Over de libratie der drie binnenste groote satellieten van Jupiter en eene nieuwe methode ter bepaling van de massa van Satelliet I."* Handelingen van het 10de Congres, pages 125—128.

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in all. For each of these corrections Δl_i to the assumed longitudes of the satellites were derived. These direct results from the observations can, however, not be used as they stand. There are, as has been mentioned above, in the longitude of each satellite four inequalities, whose periods are between 400 and 500 days, and whose coefficients are of the same order of magnitude as the libration. These inequalities therefore, during the few months over which each of the ten series of observations extends, are practically constant, and the correction Δl_i derived from the observations consequently contains, in addition to the correction $\Delta \varepsilon_i$ to the mean longitude, and the libration, also the correction to the assumed values of these inequalities.

Now the coefficients of these inequalities are proportional to the excentricities and depend on the masses, and are therefore incertain to the same extent as these, i.e. to a very large extent. The periods of the four inequalities are so nearly equal, that they cannot be separated from each other. Further the period of the most important of them — important both by its magnitude and by its uncertainty — differs just so much from the average interval of one opposition to the next that, when we consider only the values at the epochs of opposition, the inequality presents itself as one having approximately the period of the libration, and can therefore not be separated from the libration. itself. For all these reasons it was impossible to determine the libration *and* he long-periodic inequalities from these observations alone.

For the determination of the masses, leaving for the moment the mass of IV out of consideration, we have the following data:

1. the large inequalitics in the longitudes of the satellites I, II and III,
2. the motion of the perijove of satellite IV,
3. the period of the libration.

The motion of the perijove of IV also depends on the compression of the planet, which must thus also be investigated, and is determined by

4. the motion of the node of satellite II.

The data mentioned under 1, 2 and 4 are those used by LAPLACE, 3 has for the first time been pointed out by me in the communication to the "Nederlandsch Natuur- en Geneeskundig Congres", quoted above.

The method by which the approximations have been conducted is the following. Certain values of the masses, approximately verifying he conditions 1, 2, and 4, are assumed, and the corresponding in all. For each of these corrections Δl_i to the assumed longitudes of the satellites were derived. These direct results from the observations can, however, not be used as they stand. There are, as has been mentioned above, in the longitude of each satellite four inequalities, whose periods are between 400 and 500 days, and whose coefficients are of the same order of magnitude as the libration. These inequalities therefore, during the few months over which each of the ten series of observations extends, are practically constant, and the correction Δl_i derived from the observations consequently contains, in addition to the correction $\Delta \varepsilon_i$ to the mean longitude, and the libration, also the correction to the assumed values of these inequalities.

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$$\Delta l_i' = \Delta l_i - (\delta l_i' - \delta l_i^0).$$

From these $\Delta l_i'$ we then determine the amplitude, the phase and the period of the libration. If this period co-incides with the one computed from the assumed masses, then the approximation is sufficient, if not, then the whole process is repeated with different masses.

The communication of the different approximations and of the residuals remaining after the substitution of the finally adopted values, would exceed the limits set to this paper. The formula finally derived for the libration is

$$\vartheta = 0^\circ.158 \, sin \frac{t - 1895{\cdot}09}{7{\cdot}0}.$$

The adopted masses are

$$m_1 = 0.0000\ 256$$
$$m_2 = 0.0000\ 231$$
$$m_3 = 0.0000\ 820$$

and the corresponding ratio of the distribution of the libration over the longitudes of the three satellites is given by

$$\frac{\vartheta_1}{\vartheta} = +\ 0{\cdot}175 \qquad \frac{\vartheta_2}{\vartheta} = -\ 0{\cdot}260 \qquad \frac{\vartheta_3}{\vartheta} = +\ 0{\cdot}022^5$$

The mean longitudes (excluding libration) on 1900 January 0, Greenwich mean noon, are (counted from the point Aries)

$$l_1 = 142^\circ{\cdot}604$$
$$l_2 = \ \ 99\ {\cdot}534$$
$$l_3 = 167\ {\cdot}999$$
$$l_4 = 234\ {\cdot}372,$$

By a comparison of these with the values at the epoch 1750.0 the following sidereal mean daily motions [1]) were derived

$$n_1 = 203^\circ{\cdot}4889\ 5652$$
$$n_2 = 101\ {\cdot}3747\ 2411$$
$$n_3 = \ \ 50\ {\cdot}3176\ 0790^5$$
$$n_4 = \ \ 21\ {\cdot}5710\ 7132.$$

I have added no probable errors, which in the absence of the details of the observational material can only have a subjective value.

1) i.e. sidereal mean motions in a mean solar day.

$\tau_{ii} = 1$, the other ratios τ_{ij}, and the motions $\frac{d\tilde{\omega}_i}{dt}$ depending on the masses. Thus if certain values of the masses are adopted, the ratios τ_{ij} are thereby determined. If then h_i and k_i of the four satellites are known from the observations, then from the eight linear equations [13] (consisting of two sets of four each, with the same coefficients) we can determine the eight unknowns $e_i \sin \tilde{\omega}_i$ and $e_i \cos \tilde{\omega}_i$, and from these again e_i and $\tilde{\omega}_i$. The method is exactly the same as the one used by me for the determination of the inclinations and nodes (see these Proceedings, 1906 March, pages 767—780). The values of h_i and k_i have been determined from the heliometer-observations made at the Cape Observatory, in 1891 by Sir DAVID GILL, and in 1901 and 1902 by Mr. BRYAN COOKSON. The results from these observations have been treated by the method just delineated, in two different suppositions regarding the masses, i. e. regarding the ratios

Satellite	Epoch	e System I	e System II	e p.e.	$\tilde{\omega}$ System I	$\tilde{\omega}$ System II	$\tilde{\omega}$ p.e.	$\tilde{\omega}_{1900\cdot0}$ System I	$\tilde{\omega}_{1900\cdot0}$ System II
I	1891·75	0°036	0°036	±°009	158°	157°	± 15°	248°	235°
	1901·61	·055	·055	± 22	136	136	± 36	48	50
	1902·60	·022	·021	± 17	262	270	± 27	120	131
II	1891·75	0 018	0 020	±·006	169	166	± 16.	300	274
	1901·61	·020	019	± 14	318	·315	± 37	292	294
	1902·60	·026	026	± 9	302	301	± 24	261	267
	Mean	0·021	0·022					284	278
III	1891·75	0·086	0·086	±·003	179·7	179·6	± 2 0	201·4	200·4
	1901·61	·100	·101	± 9	198·2	198·1	± 5·6	193·9	193·8
	1902·60	·080	·080	± 6	219·0	218 8	± 4·0	212 2	212·3
	Mean	0·089	0·089					202·5	202·2
IV	1891·75	0·4284	0·4280	±·0015	142·28	142 29	± 0·20	148·19	147·83
	1901·61	·4228	·4216	± 30	148 92	149·05	± ·40	147·76	147·96
	1902·60	·4261	·4262	± 25	149·06	149·03	± ·34	147·20	147·28
	Mean	0·4258	0·4253					147.72	147·69

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