Mathematics. — "*The extension of the Configuration of* KUMMER *to spaces of* (2^p-1) *dimensions.*" By Mr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG.)

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§ 1. If we represent by S_1 the system $\begin{matrix} a & b \\ b & a \end{matrix}$, built up out of two letters and by S_2 the same system in new letters c and d; if likewise we represent by T the system of signs $\begin{matrix} + & + \\ + & - \end{matrix}$ and by $-T$ the opposite $\begin{matrix} - & - \\ - & + \end{matrix}$, we obtain by connecting these

$$\begin{matrix} S_1 & S_2 \\ S_2 & S_1 \end{matrix} \quad \text{and} \quad \begin{matrix} T & T \\ T & -T \end{matrix}$$

the two systems

$$\begin{matrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{matrix} \quad \text{and} \quad \begin{matrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{matrix}$$

By giving *each* row of four letters in turn the signs of *each* row of the system of signs sixteen quadruplets of algebraic quantities appear which, as is known [1]), represent the elements of the $Cf(16_6)$ of KUMMER whether they are considered as homogeneous coordinates of points or as coefficients of planes in Sp_3. For, to each element are incident the elements of another kind, represented by the three permutated letter quadruplets and for each of them with half of the sign combinations.

§ 2. If now we call S_1 and T the letter- and the sign-system of 4 resp. and if we repeat the combination described above suchlike systems of 8 are formed of which that one of the letters furnishes the permutations of a *regular* G_8 of order 8 [2]), consisting exclusively of binary substitutions, whilst that of the signs is *anallagmatic* [3]), i. e. every two rows show as many sign variations as

[1]) See a.o. JESSOP *Line-Complex* p. 23 or HUDSON *Kummer's Surface* p. 5.

[2]) Compare MILLER *Quart. Journ.* 28 p. 255, group 8 No. 4.

[3]) LUCAS *Récréations Mathématiques* II p. 113; *Nieuw Archief voor Wiskunde* 7 p. 256.

sign-permanencies. The systems become (that of the signs somewh differently arranged):

I	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
II	*b*	*a*	*d*	*c*	*f*	*e*	*h*	*g*
III	*c*	*d*	*a*	*b*	*g*	*h*	*e*	*f*
IV	*d*	*c*	*b*	*a*	*h*	*g*	*f*	*e*
V	*e*	*f*	*g*	*h*	*a*	*b*	*c*	*d*
VI	*f*	*e*	*h*	*g*	*b*	*a*	*d*	*c*
VII	*g*	*h*	*e*	*f*	*c*	*d*	*a*	*b*
VIII	*h*	*g*	*f*	*e*	*d*	*c*	*b*	*a*

1	+	+	+	+	+	+	+
2	+	+	+	+	−	−	−
3	+	+	−	−	+	+	−
4	+	−	+	−	+	−	+
5	+	−	−	+	−	+	+
6	+	−	−	+	+	−	−
7	+	−	+	−	−	+	−
8	+	+	−	−	−	−	+

By providing each of the rows of letters with each of the sigı combinations there appear sixty-four octuples of algebraic numbeı to which we assign the notations I 1, I 2, ... VIII 8. Whether w consider these numbers as homogeneous coordinates of points or ε coefficients of equations of Sp_6 in a Sp_7, each element is incident ı $7 \times 4 = 28$ of another sort, namely to half of the sign combinatioı of each letter permutation; so a *Cf.* (64_{28}) appears, to be designate by K^{VII}.

As with K^{III} it is possible to combine the *Cf*-elements to sin plexes *A, B, C, D, E, F, G, H* in various ways. Such an arrang ment is i. a.:

	1	2	3	4	5	6	7	8
A	I 1	II 4	III 5	IV 3	V 7	VI 8	VII 6	VIII 2
B	I 2	II 7	III 6	IV 8	V 4	VI 3	VII 5	VIII 1
C	I 3	II 6	III 7	IV 1	V 5	VI 2	VII 4	VIII 8
D	I 4	II 1	III 8	IV 6	V 2	VI 5	VII 3	VIII 7
E	I 5	II 8	III 1	IV 7	V 3	VI 4	VII 2	VIII 6
F	I 6	II 3	III 2	IV 4	V 8	VI 7	VII 1	VIII 5
G	I 7	II 2	III 3	IV 5	V 1	VI 6	VII 8	VIII 4
H	I 8	II 5	III 4	IV 2	V 6	VI 1	VII 7	VIII 3

The table indicates that eight vertices of e.g. the simplex *A* aı resp. the points I 1, II 4 etc, according to the former notation, whil at the same time the eight opposite side-Sp_6 of the simplex aı represented by those same notations.

The connection of Cf-elements can now be represented by a diagram (pl. I) the rows of which indicate the Sp_6, the columns the points, whilst incidence of a Sp_6 with a point is indicated by hatching the square common to the respective row and column.

We see that the diagram can be brought to a more condensed shape:

	A	B	C	D	E	F	G	H
A	S	a	b	c	d	e	f	g
B	a	S	g	f	e	d	c	b
C	b	g	S	e	f	c	d	a
D	c	f	e	S	g	b	a	d
E	d	e	f	g	S	a	b	c
F	e	d	c	b	a	S	g	f
G	f	c	d	a	b	g	S	e
H	g	b	a	d	c	f	e	S

Here S indicates a simplex-filling; each of the other letters a system (8_3) denoting the incidence connection between the elements of two simplexes. These systems (8_3) have all degenerated into two (4_3), each pair of our simplexes is thus connected in an equal way and forms a $Cf(16_{10})$ of the same type.

§ 3. Analogous to the well-known decomposition of K^{III} into four tetrahedra lying in pairs in a Möbius-position, it is obvious to call the position of two of the simplexes, e. g. A and B, by that name. Each side-Sp_6 of one S contains three points, so a face, of the other; each vertex of one lies in three side-Sp_6, so in a side-Sp_4 of the other; the correspondence is such that opposite elements of A, e.g. vertex A_1 and side-space A_1 also furnish opposite elements of B, namely resp. the side-Sp_4: $B_1B_5B_6B_7B_8$ and the face $B_2B_3B_4$, just as this is the case with the tetrahedra in Möbius-position.

There exists already however. provided with the same property, an extension of this notion, that of Berzolari [1]) where each side-Sp_6 of one S contains one vertex of the other, and is generated by operation with a focal system on an arbitrary simplex; let us call this position $M\,I$, then it is evident that the discussed more specialized $M\,II$ is to be regarded as a threefold $M\,I$.

[1]) *Rendiconti del Circolo Matem. di Palermo* 22.

The elements of two simplexes A and B in $M\,II$ can be arranged only in one other way to two suchlike simplexes, namely as

first simplex $P: A_1, A_2, A_3, A_4, B_5, B_6, B_7, B_8,$
second „ $Q: B_1, B_2, B_3, B_4, A_5, A_6, A_7, A_8.$

If we regard such a new simplex in connection with $C, D, \ldots H$, it then shows with each of these a new sort of position; for all however of the same type, showing analogy to the pairs of tetrahedra in STEINER-position which can be separated in the same way from K^{III} [1]). We find for the $cf(16_{10})$ of two such simplexes a diagram of the shape:

$$\begin{matrix} S & x \\ x & S, \end{matrix}$$

where x again represents a system (8_8) which however does not degenerate now, but is identical to the cyclic system which is obtained out of the initial row: 1 2 . . 5 . . .

Opposite elements of one simplex furnish, as in Sp_3, no opposite ones of the other.

§ 4. The 28 operations determining in each cf-space the cf-points incident to them and reciprocally, are *focal-correlations*; thus e.g. the $Sp_6 : A_1$

$$(+a, +b, +c, +d, +e, +f, +g, +h)$$

is transformed into the point A_2 situated in it

$$(+b, -a, +d, -c, +f, -e, +h, -g)$$

by operating with the skew-symmetrical determinant of transformation:

$$\begin{vmatrix} 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix}$$

These focalsystems are mutually in involution as the group of the letter substitutions as well as that of the sign variations are ABEL groups.

The 36 remaining reciprocities are polarities with respect to some 36 quadratic Sp_6, which serve for K^{VII} as the 10 fundamental-surfaces of order two for K^{III}.

[1]) MARTINETTI, *Rendic. Palermo* 16 p. 196.

Their equations are of two types; namely *eight* of the form

$$\pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_4^2 \pm x_5^2 \pm x_6^2 \pm x_7^2 \pm x_8^2 = 0\,,$$

where the combinations of signs must be derived from the sign system; and twenty-eight of the form:

$$\pm x_1 x_2 \pm x_3 x_4 \pm x_5 x_6 \pm x_7 x_8 = 0\,,$$

where the connection of the indices is given by the seven binary substitutions of the regular G_8, whilst the signs must be selected:

$$\begin{matrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{matrix}$$

The sixty-three operations which transform an element into another of the same sort are *collineations*; so we obtain, analogous to the KLEIN G_{32} in Sp_3, *a geometrical* ABEL *group* G_{128}, *consisting of the identity and sixty-three collineations; twenty-eight focal systems in involution and thirty-six polarities.*

§ 5. *The twenty-eight points in each* Sp_6 *of* K^{VII} *lie on a quadratic* Q_5 *and reciprocally.*

To prove this we regard the determinant of the terms of order two, formed of seven of the eight homogeneous coordinates; so this is of order $7 + \binom{2}{7} = 28$. The omission of a coordinate is geometrically the projecting out of a vertex of the fundamental simplex on the opposite Sp_6; if the projections of 28 points lie in it quadratically, then the points themselves do so in their Sp_6.

Let us first restrict ourselves to $Sp_6 : A_1$.

The twenty-eight points are to be divided into seven quadruplets of the same order of letters; the purely quadratic terms within such a quadruplet are in each column alike, the mixed ones may differ in sign. Let us call the four terms in a column $p,\ q,\ r,\ s$, then the substitution

$$\begin{aligned} P &= p + q + r + s \\ Q &= p + q - r - s \\ R &= p - q + r - s \\ S &= p - q - r + s \end{aligned} \quad , \text{ the } \Delta \equiv \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix} \text{ of which is } \neq 0,$$

causes three of the four quadratic terms to disappear, the Δ_{28} breaks up into the product of a Δ_7 of quadratic and a Δ_{21} of mixed terms. Here

$$\Delta_7 = \begin{vmatrix} b^2 & a^2 & d^2 & c^2 & f^2 & e^2 & h^2 \\ c^2 & d^2 & a^2 & b^2 & g^2 & h^2 & e^2 \\ d^2 & c^2 & b^2 & a^2 & h^2 & g^2 & f^2 \\ e^2 & f^2 & g^2 & h^2 & a^2 & b^2 & c^2 \\ f^2 & e^2 & h^2 & g^2 & b^2 & a^2 & d^2 \\ g^2 & h^2 & e^2 & f^2 & c^2 & d^2 & a^2 \\ h^2 & g^2 & f^2 & e^2 & d^2 & c^2 & b^2 \end{vmatrix}$$

That in general $\Delta_7 \neq 0$ is evident i. a. from

$$h = 1 \;,\; a = b = c = d = e = f = g = 0.$$

The Δ_{21} gets after change of signs of some rows the form:

0	0	0	0	bh	0	0	0	0	$-ah$	0	df	$-de$	0	$-cf$	ce	0	0	0
0	0	bf	$-be$	0	0	0	$-af$	ae	0	0	0	0	dh	0	0	$-ch$	0	0
bd	$-bc$	0	0	0	$-ad$	ac	0	0	0	0	0	0	0	0	0	0	0	fh
0	0	0	ch	0	0	0	dg	0	$-de$	0	0	$-ah$	0	$-bg$	0	be	0	0
0	0	cg	0	$-ce$	0	0	0	dh	0	0	$-ag$	0	ae	0	$-bh$	0	0	0
0	bc	0	0	0	ad	0	0	0	0	$-ab$	0	0	0	0	0	0	$-gh$	0
0	0	dh	0	0	0	0	0	cg	$-cf$	0	0	$-bg$	bf	$-ah$	0	0	0	0
0	0	0	dg	$-df$	0	0	ch	0	0	0	$-bh$	0	0	0	$-ag$	af	0	0
$-bd$	0	0	0	0	0	$-ac$	0	0	0	ab	0	0	0	0	0	0	gh	$-fh$
0	$-eh$	0	0	0	$-fg$	0	0	0	cf	0	0	bg	0	ah	0	0	0	0
$-eg$	0	0	0	ce	0	$-fh$	0	0	0	0	ag	0	0	0	bh	0	0	$-ac$
0	0	0	be	0	0	0	af	0	0	$-gh$	0	0	0	0	0	ch	$-ab$	0
$-fh$	0	0	0	0	0	$-eg$	0	0	de	0	0	ah	0	bg	0	0	0	$-bd$
0	$-fg$	0	0	df	$-eh$	0	0	0	0	0	bh	0	0	0	ag	0	0	0
0	0	$-bf$	0	0	0	0	0	$-ae$	0	gh	0	0	$-dh$	0	0	0	ab	0
0	0	0	0	0	0	0	0	0	ah	$-ef$	0	de	0	cf	0	0	$-cd$	0
0	fg	0	$-dg$	0	eh	0	$-ch$	0	0	0	0	0	0	0	0	$-af$	0	0
eg	0	$-cg$	0	0	0	fh	0	$-dh$	0	0	0	0	$-ae$	0	0	0	0	ac
0	0	0	0	$-bh$	0	0	0	0	0	ef	$-df$	0	0	0	$-ce$	0	cd	0
fh	0	0	$-ch$	0	0	eg	$-dg$	0	0	0	0	0	0	0	0	$-be$	0	bd
0	eh	$-dh$	0	0	fg	0	0	$-cg$	0	0	0	0	$-bf$	0	0	0	0	0

The sum of the numbers in each column amounts to zero; so

$$\Delta_{21} = 0.$$

As each element with the 28 incident to it can be transformed into any other by means of a direct or reciprocal projectivity, the quadratic position of every 28 is now proved.

§ 6. Each couple of Sp_6 of the *cf* has twelve points in common lying thus in a Sp_5. No other Sp_6 containing these twelve, all these Sp_5 differ and their number is $\binom{2}{64} = 2016$. The *cf*-points form with them a *cf* $(64_{378}, 2016_{12})$.

There are triplets of Sp_6 which have six points in common, lying thus in a Sp_4, each *cf*-Sp_5 has namely in still 32 Sp_6 *six* of its points. Such a sextuple can be deduced from three groups of twelve, their number is thus $\frac{2016 \times 32}{3} = 21504$; they form with the *cf*-Sp_5 a *cf* $(21504_3, 2016_{32})$.

There are quadruplets of Sp_6 having four points in common which therefore determine a Sp_3; each *cf*-Sp_4 has namely four of its six points in fifteen other *cf*-Sp_6. Every Sp_3 can be derived from four Sp_4, their number is thus $\frac{21504 \times 15}{4} = 80640$. They form with the *cf* Sp_4 a *cf* $(80640_4, 21504_{15})$.

There are sextuplets of Sp_6 having three points of the *cf* in common, which therefore determine a Sp_2; each *cf*-Sp_3 has namely three of its four points in eight other *cf*-Sp_6 more, these eight Sp_6 furnish two by two however the same triplet; as furthermore each Sp_2 can be deduced from $\binom{2}{6} = 15$ Sp_3 their number is $\frac{80640 \times 4}{15} = 21504$. This could be expected as the whole consideration starting from the *cf*-points might have been put reciprocally, and would then have led on account of the self-reciprocity of the system to the same elements; so still 2016 Sp_1 are obtained, the right lines of connection of the pairs of points.

The further amounts of incidences of the kinds of elements mutually can now be easily deduced; the notation of K^{VII} becomes finally:

	Sp_0	Sp_1	Sp_2	Sp_3	Sp_4	Sp_5	Sp_6
	64	2016	21504	80640	21504	2016	64
incident to:							
Sp_0	—	2	3	4	6	12	28
Sp_1	63	—	3	6	15	66	378
Sp_2	1008	32	—	4	21	160	2016
Sp_3	5040	240	15	—	15	240	5040
Sp_4	2016	160	21	4	—	32	1008
Sp_5	378	66	15	6	3	—	63
Sp_6	28	12	6	4	3	2	—

By the method of intersecting and projecting triplets and doublets of consecutive kinds of elements are to be transformed into elements of Sp_3 or Sp_2; thus are formed e.g. a *cf* (21504_{21}) of points and planes, with 80640 *cf*-lines, and a plane *cf* $(2016_{32}, 21504_3)$ of points and lines, or reciprocally.

§ 7. If we represent the system of letters and that of signs of 8 resp. by S_1 and T and if we repeat the combination

$$\begin{matrix} S_1 & S_2 \\ S_2 & S_1 \end{matrix} \quad \text{and} \quad \begin{matrix} T & T \\ T & -T, \end{matrix}$$

we obtain systems for 16 belonging to each other, etc., the operation allowing of indefinite continuation; one always arrives at a regular ABEL substitution group G_{2^p} and a suitable anallagmatical system for the signs.

These always furnish in R_{2^p-1} a *cf*, analogous to that of KUMMER with the notation:

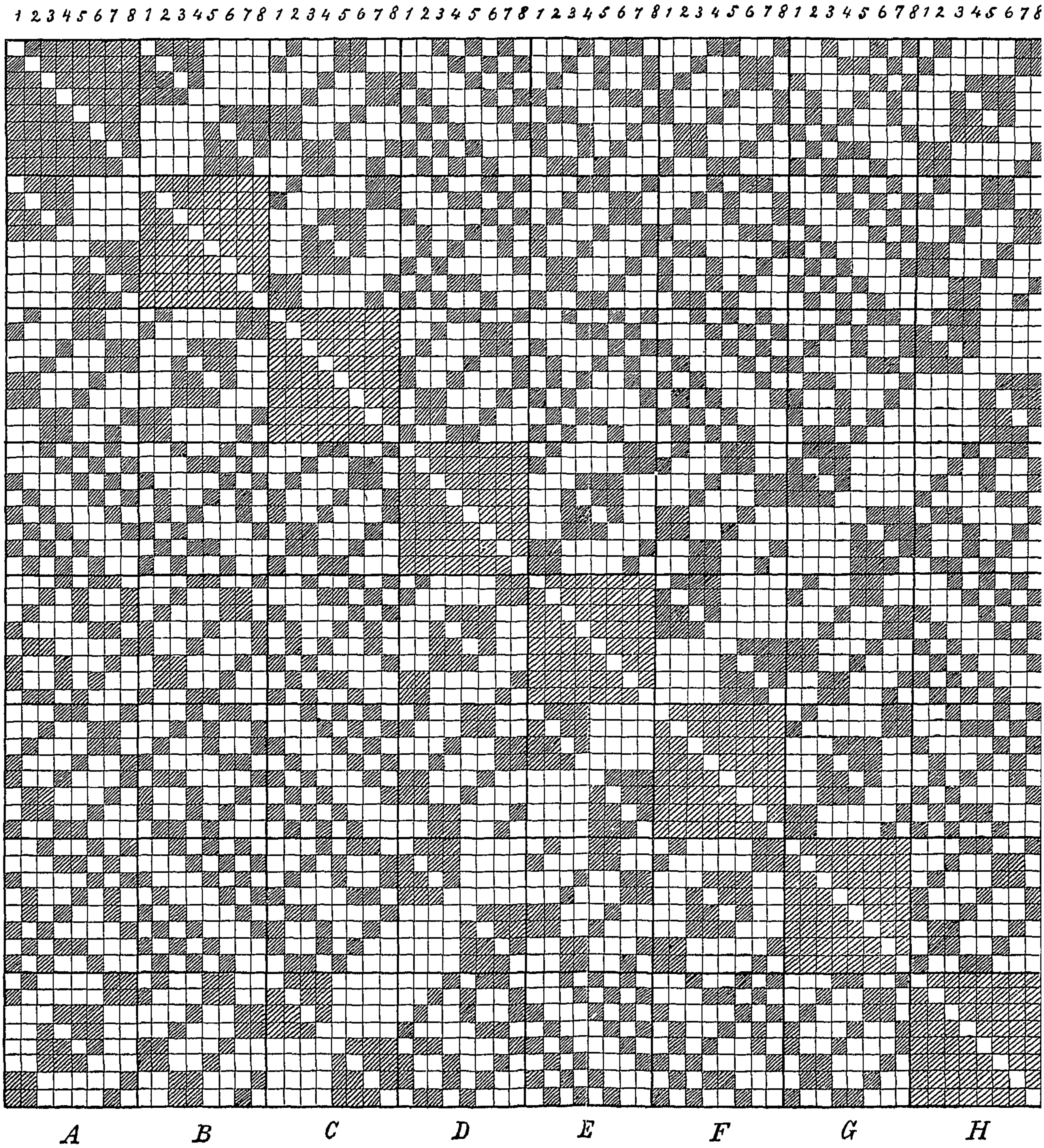
$$Cf\left(2^{2p}_{(2^p-1)\,.\,2^{p-1}}\right),$$

arising from an arbitrary starting element by an operation with a geometrical ABEL group:

the *identity* and $2^{2p}-1$ *collineations* on one hand
and $(2^p-1)\,.\,2^{p-1}$ *focal systems* mutually in involution with
$(2^{p+1}-2^p+1)\,.\,2^{p-1}$ *polarities* on the other hand.

The quadratic situation of the elements incident to one element can always be proved by reduction of the determinant according to the example of § 5 [1]).

[1]) A more extensive treatment also for spaces of other numbers of dimensions will follow in the dissertation to be published: J. A. BARRAU, *Bydragen tot de theorie der cff.* (Amsterdam 1907).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A B C D E F G H