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F. Fornix.
L.c. Limbus corticalis.
L.m. Limbus medullaris.

V.l. Lateral ventricle.
V.t. Third ventricle.

Fig. II.

Frontal section through the more posterior part of the corpus callosum. Section 15 μ stained with haematoxylin and eosin. Enl. 13 diam.

C. Zone of union of the pallia.
C.a. Anterior commissure.
C.c. Corpus callosum.
C.ch. Corpus chorioideum.
C.str. Corpus striatum.
F. Fornix.

L.c. Limbus corticalis.
L.m. Limbus medullaris.
L.tr. Lamina trapezoidea.
Ps. Fornix commissure.
V.l. Lateral ventricle.
V.t. Third ventricle.

Mathematics. — “*On an infinite product, represented by a definite integral.*” By Prof. W. KAPTEYN.

The object of this paper is to write the infinite product

$$\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right)$$

in the form of a definite integral.

This product is connected with *mod.* $\Gamma(u+iv)$, for

$$\text{mod. } \Gamma(u+iv) = \Gamma(u) \cdot e^{-P(u,v)} (u > 0)$$

where

$$P(u,v) = \frac{1}{2} \sum_{s=0}^{\infty} \lg \left(1 + \frac{v^2}{(u+s)^2} \right)^{-1})$$

thus

$$\text{mod. } \Gamma(u+iv) = \frac{\Gamma^2(u)}{\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right)}.$$

and

$$\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right) = \frac{\Gamma^2(u)}{\text{mod. } \Gamma(u+iv)}.$$

To write the second member of this equation in the form of a definite integral, we start from WEIERSTRASS' definition

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_W e^{t-z} dt$$

where the integral is taken along a curve W commencing at negative

¹⁾ Nielsen. Handbuch der Theorie der Gammafunctionen p. 23.

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infinity, circulating around the origin in the positive direction, and returning to negative infinity again; thus

$$\frac{2\pi i}{\Gamma(z)} = e^{\pi iz} \int_0^\infty e^{-t} t^{-z} dt - e^{-\pi iz} \int_0^\infty e^{-t} t^{-z} dt$$

and if $z = u + iv$

$$\begin{aligned} \frac{2\pi i}{\Gamma(u+iv)} &= e^{\pi u} \left(\cos(\pi u) + i \sin(\pi u) \right) \int_0^\infty e^{-t} t^{-u} \left(\cos(v \lg t) - i \sin(v \lg t) \right) dt \\ &\quad - e^{\pi u} \left(\cos(\pi u) - i \sin(\pi u) \right) \int_0^\infty e^{-t} t^{-u} \left(\cos(v \lg t) + i \sin(v \lg t) \right) dt. \end{aligned}$$

Writing

$$\begin{aligned} \int_0^\infty e^{-t} t^{-u} \cos(v \lg t) dt &= M \\ \int_0^\infty e^{-t} t^{-u} \sin(v \lg t) dt &= N \\ \frac{2\pi}{\Gamma(u+iv)} &= \alpha + i\beta \end{aligned}$$

we obtain

$$\alpha = (e^{\pi u} + e^{-\pi u}) \sin(\pi u) M + (e^{\pi u} - e^{-\pi u}) \cos(\pi u) N$$

$$\beta = (e^{\pi u} - e^{-\pi u}) \cos(\pi u) M - (e^{\pi u} + e^{-\pi u}) \sin(\pi u) N$$

and

$$\alpha^2 + \beta^2 = (e^{2\pi u} - 2 \cos 2\pi u + e^{-2\pi u})(M^2 + N^2).$$

Now we have

$$\begin{aligned} M^2 &= \int_0^\infty e^{-x} x^{-u} \cos(v \lg x) dx \cdot \int_0^\infty e^{-y} y^{-u} \cos(v \lg y) dy \\ N^2 &= \int_0^\infty e^{-x} x^{-u} \sin(v \lg x) dx \cdot \int_0^\infty e^{-y} y^{-u} \sin(v \lg y) dy \end{aligned}$$

so

$$M^2 + N^2 = \int_0^\infty \int_0^\infty e^{-(x+y)} (xy)^{-u} \cos\left(v \lg \frac{y}{x}\right) dx dy$$

or in polar coordinates, putting

$$x = r \cos \theta, y = r \sin \theta$$

$$M^2 + N^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r(\cos \theta + \sin \theta)} (r^2 \sin \theta \cos \theta)^{-u} \cos(v \lg r \tan \theta) r dr d\theta.$$

This double integral may be reduced to a single one, for

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$$\int_0^\infty e^{-r} (\cos \theta + \sin \theta) r^{-2u+1} dr = \frac{\Gamma(2-2u)}{(\cos \theta + \sin \theta)^{2-2u}} (u < 1)$$

therefore

$$M^2 + N^2 = \Gamma(2-2u) \int_0^{\frac{\pi}{2}} \cos(v \lg \operatorname{tg} \theta) \frac{(\sin \theta \cos \theta)^{-u}}{(\cos \theta + \sin \theta)^{2-2u}} d\theta$$

or

$$M^2 + N^2 = 2\Gamma(2-2u) \int_0^{\frac{\pi}{4}} \cos(v \lg \operatorname{tg} \theta) \frac{(\sin \theta \cos \theta)^{-u}}{(\cos \theta + \sin \theta)^{2-2u}} d\theta.$$

If in this integral, we change the variable by the substitution

$$\operatorname{tg} \theta = e^{-2t}$$

it takes the form:

$$M^2 + N^2 = 4 \Gamma(2-2u) \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2-2u}}$$

With this value we find

$$\frac{4\pi}{\operatorname{mod}^2 \Gamma(u+i\nu)} = 4 \Gamma(2-2u) (e^{2\pi\nu} - 2 \cos 2\pi u + e^{-2\pi\nu}) \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2-2u}}$$

and finally

$$\prod_{s=0}^{\infty} \left(1 + \frac{v^2}{(u+s)^2} \right) = \frac{\Gamma^2(u) \Gamma(2-2u)}{\pi^2} (e^{2\pi\nu} - 2 \cos 2\pi u + e^{-2\pi\nu}) \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2-2u}}$$

which holds for all values of v , and for values of u between 0 and 1.

If for instance we put $v = \frac{z}{2\pi}$, $u = \frac{1}{4}$ and $\frac{3}{4}$ we obtain

$$\begin{aligned} \left(1 + \frac{4z^2}{\pi^2} \right) \left(1 + \frac{4z^2}{25\pi^2} \right) \left(1 + \frac{4z^2}{81\pi^2} \right) \dots &= \\ &= \frac{\Gamma^2\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{2}\right)}{\pi^2} (e^z + e^{-z}) \int_0^\infty \frac{\cos\left(\frac{zt}{\pi}\right) dt}{(e^t + e^{-t})^{\frac{3}{2}}} \end{aligned}$$

and

$$\begin{aligned} \left(1 + \frac{4z^2}{9\pi^2} \right) \left(1 + \frac{4z^2}{49\pi^2} \right) \left(1 + \frac{4z^2}{121\pi^2} \right) \dots &= \\ &= \frac{\Gamma^2\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right)}{\pi^2} (e^z + e^{-z}) \int_0^\infty \frac{\cos\left(\frac{zt}{\pi}\right) dt}{(e^t + e^{-t})^{\frac{1}{2}}} \end{aligned}$$

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Writing $u = 1 - u'$ we may also conclude from the preceding that

$$\frac{1}{\text{mod}^2 \Gamma(1-u'+iv)} = \frac{\Gamma(2u')}{\pi^2} (e^{2\pi v} - 2 \cos 2\pi u' + e^{-2\pi v}) \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{u'}}$$

or, because

$$\Gamma(u'+iv) \Gamma(1-u'+iv) = \frac{\pi}{\sin \pi (u'+iv)}$$

$$\text{mod}^2 \Gamma(u'+iv) = 4 \Gamma(2u') \int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u'}}$$

which formula holds not only for $0 < u' < 1$, but also for $u' > 1$.

Introducing in this equation, the infinite product, we have

$$\int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u'}} = \frac{\Gamma^2(u')}{4 \Gamma(2u')} \frac{1}{\prod_{s=0}^\infty \left(1 + \frac{v^2}{(u'+s)^2}\right)}$$

a formula which enables us to write the integral in a finite form in two cases viz. $u' = n$ and $u' = n - \frac{1}{2}$. If $u' = n =$ positive number

$$\prod_{s=0}^\infty \left(1 + \frac{v^2}{(n+s)^2}\right) = \prod_{s=n}^\infty \left(1 + \frac{v^2}{s^2}\right);$$

with

$$\frac{e^{\pi v} - e^{-\pi v}}{2\pi v} = \prod_{s=1}^\infty \left(1 + \frac{v^2}{s^2}\right)$$

this gives

$$\int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2u}} = \frac{\pi v \Gamma^2(n)}{2 \Gamma(2n)} \frac{\prod_{s=1}^{n-1} \left(1 + \frac{v^2}{s^2}\right)}{\frac{e^{\pi v} - e^{-\pi v}}{2\pi v}}.$$

If $u' = n - \frac{1}{2}$, we have

$$\prod_{s=0}^\infty \left(1 + \frac{v^2}{(n - \frac{1}{2} + s)^2}\right) = \prod_{s=n-1}^\infty \left(1 + \frac{v^2}{(\frac{1}{2} + s)^2}\right)$$

which gives with

$$\frac{e^{\pi v} + e^{-\pi v}}{2} = \prod_{s=0}^\infty \left(1 + \frac{v^2}{(\frac{1}{2}+s)^2}\right)$$

this result

$$\int_0^\infty \frac{\cos(2vt) dt}{(e^t + e^{-t})^{2n-1}} = \frac{\Gamma^2(n - \frac{1}{2})}{2 \Gamma(2n - 1)} \cdot \frac{\prod_{s=0}^{n-2} \left(1 + \frac{v^2}{(\frac{1}{2} + s)^2}\right)}{\frac{e^{\pi v} + e^{-\pi v}}{2}}$$