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Crystallography. — "*On the permissible orders of the axes of symmetry in crystallography.*" By Prof. W. VOIGT at Göttingen. (Communicated by H. A. LORENTZ).

(Communicated in the meeting of November 30, 1907.)

In one of the articles of the second part of his collected papers, H. A. LORENTZ took up the question — which is equally important in both crystallography and crystalphysics — of the permissible order of an axis of symmetry of the first or the second kind. In this investigation he proceeds from the principle of the rational duplicate ratio, from which he first proves that it is consistent with itself, and therefore a suitable basis for crystallographical deductions.

The study of this interesting treatise led me to the thought, that for the purpose at hand another fundamental principle of crystallography — viz. that of the rational indices — might well form a more convenient starting point. The continuation of this thought led me to the following development, which, I believe, attains the end in view in a remarkably simple and short manner. I will prove for this useful fundamental principle, as LORENTZ did for the principle of the rational duplicate ratio, that it does not contradict itself and then derive from it the permissible orders of the axes of symmetry.

1. The principle of the rational indices, as is well known, is as follows.

If we select three arbitrary boundary surfaces of a crystal polyhedron and draw through any point O parallels to their lines of intersection to form a system of axes; if we choose further two other arbitrary positions through this system of axes, and then the intercepts of these planes upon the axes are

$$u = OA,\quad v = OB,\quad w = OC \quad \text{on the one hand,}$$
$$u' = OA',\quad v' = OB',\quad w' = OC' \quad \text{on the other,}$$

the principle of the rational indices maintains then, that,

$$\frac{u'}{u} : \frac{v'}{v} : \frac{w'}{w} = z_1 : z_2 : z_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

forms at all times a ratio of whole numbers.

In order that this principle should lead to no contradiction, it is necessary that if one proceeds from three *other* boundary surfaces of the polyhedron and uses *their* lines of intersection as the fundamental system of axes, then the polyhedral surfaces have *also on these axes* intercepts with the above mentioned relation, if the principle held with reference to the first system of axes.

The following consideration with reference to figure 1 proves that

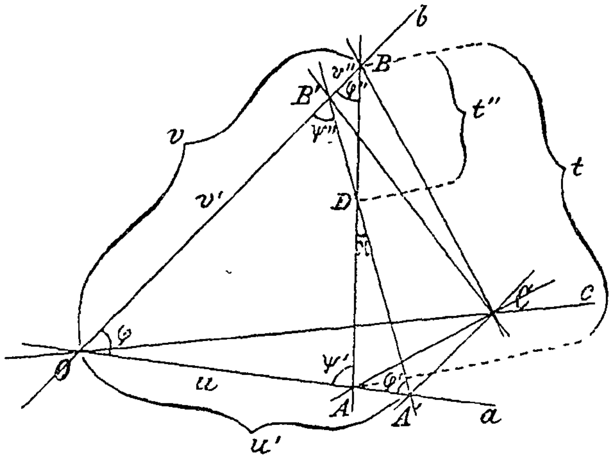

Fig. 1.

this is true. *Oa, Ob, Oc* form the first fundamental system of axes; *ABC* and *A'B'C* represent the two planes for which the principle of the rational indices holds, i.e. their intercepts on the axes fulfil equation (1). For a simple proof it is essential that we let the two planes cut the *Oc*-axis in the same point, so that $w = w'$ and equation (1) assumes the form

$$\frac{u'}{u} : \frac{v'}{v} : 1 = z_1 : z_2 : z_3 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As second system of axes we take the lines *BO, BA* and *BC*, and as second pair of surfaces, which likewise cut the *BC*-axis in one point, the surfaces *A'B'C* and *AOC*. If the principle is to lead to no contradiction, from (2) must follow

$$\frac{t''}{t} : \frac{v''}{v} : 1 = \zeta_1 : \zeta_2 : \zeta_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which the notation to the left is explained by the figure and ζ_1, ζ_2 and ζ_3 are likewise whole numbers.

If we understand by r and ϱ rational fractions we can expect that

$$\frac{u'}{u} : \frac{v'}{v} = r \text{ may follow from } \frac{t''}{t} : \frac{v''}{v} = \varrho,$$

while at the same time

$$\frac{v'}{v} = r' \text{ follows from } \frac{v''}{v} = \varrho'.$$

The latter is apparent; for by the figure $v' + v'' = v$ and hence $r' + \varrho' = 1$, follows; consequently if r' is rational, then ϱ' is also rational.

For the former the proof follows by repeated application of the law of sines, which gives according to the figure

$$\frac{u}{\sin\varphi''}=\frac{v}{\sin\psi'}=\frac{t}{\sin\varphi},\quad \frac{u'}{\sin\psi''}=\frac{v'}{\sin\varphi'},\quad \frac{v''}{\sin\chi}=\frac{t''}{\sin\psi''},$$

while

$$\chi=\psi'-\varphi'=\psi''-\varphi'',\quad \pi=\varphi+\psi'+\varphi''=\varphi+\varphi'+\psi''.$$

From these we get

$$r=\frac{\sin(\varphi+\varphi')\sin(\varphi+\varphi'')}{\sin\varphi'\sin\varphi''},\quad \varrho=\frac{\sin(\varphi+\varphi')\sin(\varphi+\varphi'')}{\sin(\varphi+\varphi'+\varphi'')\sin\varphi}.$$

The relation between r and ϱ, is most easily obtained by determining φ' from the first formula and substituting this value in the second. We thus obtain

$$r/(r-1)=\varrho.$$

This shows that a rational r leads to a rational ϱ, which was to be proved.

The last part of the proof can still be simplified, according to a suggestion by LORENTZ, if we assume the MENELAUS' Theorem as known.

The desired proof is also given, when from

$$\frac{v'}{v}=r' \text{ and } \frac{u'}{u}=r''$$

$$\frac{v''}{v}=\varrho' \text{ and } \frac{t''}{t}=\varrho''.$$

follow.

The former of these we have considered above; relative to the latter, MENELAUS Theorem gives according to the figure

$$\frac{BD}{AD}=\frac{OA'}{AA'}\cdot\frac{BB'}{OB'},$$

i. e.

$$\frac{t''}{t-t''}=\frac{u'}{u'-u}:\frac{v-v'}{v'}.$$

Hence the rationality of u'/u and v'/v gives directly the rationality of t''/t.

2. The determination of the permissible order n of an axis of symmetry follows from any one of the fundamental principles of Crystallography, but only for the case when $n \geq 5$, because each of these principles places five similar crystallographic elements in relation. We usually so proceed, that the general property which the principle gives for the cases $n \geq 5$ is also demanded for the cases $n < 5$ We can however for the latter *limited* number of cases rely upon

experience, and apply the principle only for the former *unlimited* number of cases.

Since the principle of the rational indices permits surfaces of the crystal polyhedron to be translated parallel to themselves, therefore for its application axes of symmetry of the first and second kind have exactly the same value. A difference lies only in that for axes of the second kind n must necessarily be an even number.

We start with a construction upon a sphere of unit radius, through the center of which we lay all the directions that come into consideration. (Fig. 2). Let A be trace of the n-fold axis, $P_1, P_2 \ldots P_5$ the traces of the normals of 5 related surfaces (1), (2), ... (5) of the polyhedron, such that $\psi = 2\pi/n$. The P_h's are then designated as the *poles* of these surfaces.

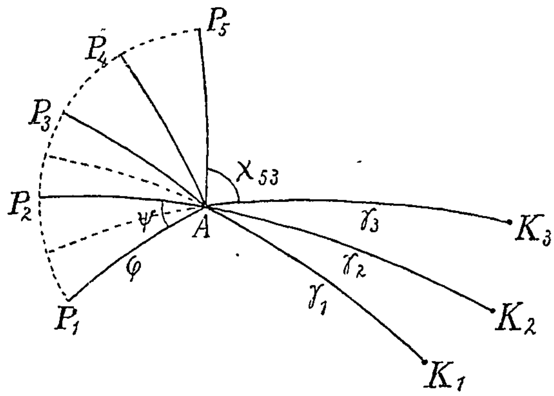

Fig. 2-

Let K_1, K_2, K_3, be the traces of the lines of intersection of the surfaces (2, 3), (3, 1), (1, 2), so that

$$K_1P_2 = K_1P_3 = \tfrac{1}{2}\pi, \quad K_2P_3 = K_2P_1 = \tfrac{1}{2}\pi, \quad K_3P_1 = K_3P_2 = \tfrac{1}{2}\pi.$$

Let K_1, K_2, K_3 form the system of axes, and (4) and (5) the pair of surfaces for the application of the principle of rational indices. It is now a question of determining the intercepts which the surfaces (4) und (5) make upon the edges K_i.

If we give the surfaces such positions that they are tangent to the sphere in their poles, then the sections σ_{hi} are identical with the reciprocals of the $\cos(K_i\ P_h)$ where $i = 1, 2, 3,\ h = 4, 5$. Consequently the values of these cosines are to be determined.

If we write γ_i, instead of AK_i, φ for AP_h and χ_{hi} for the $\angle\ P_h A K_i$ then from the $\Delta K_i A P_h$ in the figure we get

$$\cos(K\ P_h) = \cos\varphi\cos\gamma_i + \sin\varphi\sin\gamma_i\cos\chi^{hi}; \qquad . \ . \ (4)$$

further, from the ΔP_3AK_1, P_3AK_2, P_1AK_3 we get

$$tg\,\gamma_1 = tg\,\gamma_3 = \frac{ctg\,\varphi}{cos\,\frac{1}{2}\psi}\,,\ tg\,\gamma_2 = \frac{ctg\,\varphi}{cos\,\psi}\,, \quad \ldots\ldots \quad (5)$$

while direct from the figure we get

$$\left.\begin{aligned} \chi_{41} &= \pi - \frac{3}{2}\psi, \quad \chi_{42} = \pi - 2\psi, \quad \chi_{43} = \pi - \frac{5}{2}\psi, \\ \chi_{51} &= \pi - \frac{5}{2}\psi, \quad \chi_{52} = \pi - 3\psi, \quad \chi_{53} = \pi - \frac{7}{2}\psi. \end{aligned}\right\} \quad \ldots\ldots \quad (6)$$

If we write the principle of rational indices

$$\frac{\delta_{41}}{\delta_{51}} : \frac{\delta_{42}}{\delta_{52}} : \frac{\delta_{43}}{\delta_{53}} = z_1 : z_2 : z_3, \quad \ldots\ldots \quad (7)$$

and observe that in the quotients of each two δ_{hi}'s, since

$$1/\delta_{hi} = cos\,(K_i P_h) = cos\,\gamma_i cos\,\varphi\,(1 + tg\,\varphi\,tg\,\gamma_i\,cos\,\chi_{hi})$$

the factor standing before the brackets always cancel, then we can easily introduce the values (5) and (6) and obtain from (7)

$$\frac{cos\,\frac{1}{2}\psi - cos\,\frac{5}{2}\psi}{cos\,\frac{1}{2}\psi - cos\,\frac{3}{2}\psi} : \frac{cos\,\psi - cos\,3\psi}{cos\,\psi - cos\,2\psi} : \frac{cos\,\frac{1}{2}\psi - cos\,\frac{7}{2}\psi}{cos\,\frac{1}{2}\psi - cos\,\frac{5}{2}\psi} = z_1 : z_2 : z_3.$$

This gives directly

$$\frac{sin\,\frac{3}{2}\psi}{sin\,\frac{1}{2}\psi} : \frac{sin\,2\psi\,sin\,\psi}{sin\,\frac{3}{2}\psi\,sin\,\frac{1}{2}\psi} : \frac{sin\,2\psi}{sin\,\psi} = z_1 : z_2 : z_3 \quad \ldots \quad (8)$$

If we now take the first and last members of this double proportion we have

$$\frac{sin\,\frac{3}{2}\psi}{sin\,\frac{1}{2}\psi} : \frac{sin\,2\psi}{sin\,\psi} = r$$

i. e. equal to a rational fraction, or also

$$\frac{1 + 2\,cos\,\psi}{2\,cos\,\psi} = r \ \text{ i. e. } \ cos\,\psi = \frac{1}{2(r-1)} = r',$$

where r' is also rational.

This requirement, when $\psi = 2\pi/n$ and $n \geqq 5$ is fulfilled only for $n = 6$.

If we introduce this value of ψ in equation (8) we get

$$2 : \frac{3}{2} : 1 = z_1 : z_2 : z_3 ;$$

which is entirely consistent with the double proportion.

Consequently $n = 6$ is the only value ≥ 5 that is consistent with the principle of the rational indices. If we extend the requirement that $\cos \psi$ must be rational to the case $n < 5$ then the values $n = 2$, 3, 4, are also permissible for axes of the first kind, and the values $n = 2$, 4 for axes of the second kind.

Göttingen, November 1907.

Physics. — "*Isotherms of diatomic gases and their binary mixtures. VI. Isotherms of hydrogen between —104° C. and —217° C.*" (Continued). By Prof. H. KAMERLINGH ONNES and C. BRAAK. Communication N°. 100^a from the Physical Laboratory at Leiden.

(Communicated in the meeting of November 30, 1907).

§ 17. *Survey of the determinations. Remark on the apparatus.*

The measurements mentioned in this Communication comprise in the first place the supplementary determinations to which we already alluded in § 14 of Comm. N°. 99^a (Sept. 1907). These are three determinations at — 217° at a density of about 170 times the normal one.

The obvious thing to do further was to repeat the other determinations of series II with the same piezometer arranged for the determinations mentioned above, this piezometer being one of about the same dimensions as that of series II of Comm. N°. 97^a (March 1907). As a matter of fact a comparison of the values of pv_A obtained in this series with those yielded by the series III and IV teaches that the former lie somewhat, though only slightly, lower than the latter. This may be due to a systematical error as the filling in the later series was accomplished with more precautions (compare § 5 of Comm. N°. 97^a). In the series now given, just as in series IV, distilled hydrogen was used.

Both the steel tubes on the stem of the piezometer and those on the stem of the piezometer reservoir were soldered to the glass (cf. § 15 of Comm. N°. 99^a). This ensures a gas-proof connection with the steel capillary. With sealing wax it is difficult to make the connection gas-proof, because sometimes the nut begins to slide off when the flange is tightly screwed on.