Citation:

Boole Stott, A. & P.H. Schoute, On five pairs of four-dimensional cells derived from one and the same source, in:
KNAW, Proceedings, 10 II, 1907-1908, Amsterdam, 1908, pp. 499-503

$$\frac{BN}{MB} = \frac{1}{2p+1}.$$

So the theorem holds:

"If we describe in the spaces Sp_p bearing the bounding simplexes $S_{p+1}\left(\frac{2p+1}{2}\sqrt{2}\right)$ of a regular simplex $S_n\left(\frac{2p+1}{2}\sqrt{2}\right)$ simplexes $S_{p+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(p+1)\binom{n}{p+1}$ vertices of a $\frac{2p+1}{2n}(M_n)$."

For odd $n = 2n'+1$ we have in particular:

"If we describe in the spaces $Sp_{n'}$, bearing the bounding simplexes $S_{n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ of a regular simplex $S_{2n'+1}\left(\frac{2n'+1}{2}\sqrt{2}\right)$ simplexes $S_{n'+1}\left(\frac{1}{2}\sqrt{2}\right)$ concentric and oppositely orientated to the original ones we find the $(n'+1)\binom{2n'+1}{n'+1}$ vertices of a $D_{2n'}$."

In connection with the results found above the length $\frac{1}{2}\sqrt{2}$ appearing here for the edges of the new simplexes contains a confirmation.

Mathematics. — "*On five pairs of four-dimensional cells derived from one and the same source.*" By Mrs. A. Boole Stott and Prof. P. H. Schoute.

(Communicated in the meeting of December 28, 1907).

Introduction.

As this paper must be regarded as a short completion of the handbook of the "Mehrdimensionale Geometrie" included in the Sammlung Schubert we keep the notation used there.

We regard in succession each of the six regular cells C_5, C_8, C_{16}, C_{24}, C_{120}, C_{600} of the space Sp_4 and derive from these two new four-dimensional cells. The first, which has the centres K_0 of the edges of the regular cell as vertices is formed by a regular truncation at the vertices as far as the centres of the edges; the second is the reciprocal polar of the first with respect to the spherical space of the points K_0.

Because the regular C_{16} leads us to find the regular C_{24}, the number of pairs of new cells is not six but five.

I. *General observations.*

1. If we understand for the regular cells by e, k, f, r successively the number of the vertices, edges, faces, bounding bodies, by p, q the number of bounding bodies through an edge, through a point, by e', k', f' the number of vertices, edges, faces of the bounding bodies, then besides the relations

$$e + f = k + r \quad , \quad e' + f' = k' + 2$$

of EULER the equations hold

$$qe = re' \quad , \quad pk = rk' \quad , \quad 2f = rf' \ ,$$

out of which number of five we can easily deduce the relation

$$(q - 2)\, e = (p - 2)\, k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

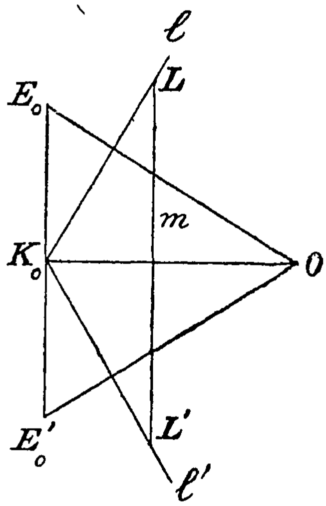
The following table furnishes these quantities for the six regular cells of Sp_4.

e	k	f	r	p	q	e'	k'	f'
5	10	10	5	3	4	4	6	4
8	24	32	16	4	8	4	6	4
120	720	1200	600	5	20	4	6	4
16	32	24	8	3	4	8	12	6
24	96	96	24	3	6	6	12	8
600	1200	720	120	3	4	20	30	12

2. We shall now endeavour to express the characteristic numbers E, K, F, R of the first of the two new cells — and what is also possible for these P, Q — in the characteristic numbers e, k, f, r, p, q of the regular cell.

"The number of vertices of the new cell is k, i. e. $E = k$."

If we project the regular cell (see the diagram) on the plane through one of the edges E_0E_0' and the centre O, the two new bounding spaces passing through the centre K_0 project themselves according to the perpendiculars l, l' let down out of K_0 on the axes OE_0, OE_0'. The section of the regular cell with a plane normal to the plane of projection in a point lying close to E_0E_0' being an equilateral triangle, a square or a regular pentagon, with the assumed point always as centre, according to p having the value 3, 4 or 5, the section of the system of the $p+2$ bounding spaces of the new cell passing through K_0 with a space normal to OK_0 — e.g. with a space which according to the normal m to OK_0 is perpendicular to the plane of projection -- is a right p-lateral prism, of which the segment LL' of m enclosed between l, l' is the axis and the perpendicular endplanes project themselves in L and L'. From this ensues:



"Through a vertex pass $p+2$ bounding spaces, i.e. $Q = p + 2$".

"Through an edge pass three bounding spaces, i.e. $P = 3$".

"Through a vertex pass $2p$ edges, so pk is the number of edges, i.e. $K = pk$."

"The system of the bounding spaces consists of two groups, namely of e regular polyhedra with q faces, and r semi-regular polyhedra (e', k', f') with equivalent vertices truncated at the vertices as far as the centres of the edges, i.e. $R = e + r$".

"As the polyhedra of the second group have $e' + f' = k' + 2$ faces and a face is common to two bounding spaces, the number of faces is half the sum of qe and $r(k' + 2)$ or qe and $pk + 2r$, i.e. $2F = qe + pk + 2r$".

Thus the result is:

"The first of the two cells, (E, K, F, R, P, Q), deduced out of the regular cell (e, k, f, r, p, q) has the characteristic numbers

$$E = k, \quad K = pk, \quad F = \tfrac{1}{2}(qe + pk) + r, \quad R = e + r,$$
$$P = 3, \quad Q = p + 2."$$

Here the law of EULER $E + F = K + R$ may serve as verification. In reality the difference of the two members of this equation

$$\begin{aligned} E + F - (K + R) &= k + \tfrac{1}{2}(qe + pk) + r - (pk + e + r) \\ &= k - e + \tfrac{1}{2}(qe - pk) \\ &= \tfrac{1}{2}\{(q-2)\,e - (p-2)\,k\} \end{aligned}$$

is equal to zero in consequence of the relation (1).

3. The second of the new cells deduced out of the regular cell is enclosed by the polar-spaces of the centres K_0 of the edges with respect to the spherical space through those points, i.e. by the

34*

tangential spaces to that spherical space in those points, i. e. by the spaces in the points K_0 normal to the axes OK_0[1]). By polar inversion of the result found above we arrive with respect to this second new cell at the following results:

"The number of bounding spaces of the new cell is k, i. e. $R' = k$."

"The bounding bodies have $p + 2$ vertices and are double pyramids with a regular polygon with p-sides as base lying in a plane bisecting the connecting line of the vertices at right angles."

"The faces are isosceles triangles."

"In a bounding space lie $2p$ faces, so pk is the number of faces, i. e. $F' = pk$."

"The system of the vertices consists of two groups, namely of e regular vertices and r semi-regular vertices, i. e. $E' = e + r$".

"The number of edges K' is $\frac{1}{2}(qe + pk) + r$."

So the result is:

"The second of the two cells, (E', K', F', R'), deduced out of the regular cell (e, k, f, r, p, q) is bounded by double pyramids with a regular polygon with p-sides as base and has the characteristic numbers

$$E' = e + r, \quad K' = \tfrac{1}{2}(qe + pk) + r, \quad F' = pk, \quad R' = k."$$

It might appear as if it were possible to deduce more pairs of new cells out of the regular cells by doing for the ends F_0 of the axes OF_0 the same as has been done above for the points K_0. This is, however, not the case, because for each regular cell the centres F_0 of the faces form the centres K_0 of the edges of another regular cell which is for the cells C_5, C_{24} dualistically related to themselves a cell of the same kind, for the cells related in pairs to one another (C_8, C_{16}) (C_{120}, C_{600}) a cell dualistically related. And as is immediately evident, the pointgroups E_0 and R_0 can neither lead to new results.

We conclude these general observations with the remark that the two cells deduced from the regular cell (e, k, f, r) show much regularity; of the former the vertices and the edges, of the latter the faces and the bounding bodies are mutually equivalent groups of elements, whilst the faces and the bounding bodies of the former and the vertices and edges of the latter form groups of elements consisting of two subgroups. Do these new cells furnish the maximum amount of regularity for polytopes not entirely regular? We do not intend to go into further details here, as the Mathematical Society at Amsterdam is proposing a prizequestion about what is to be understood by "semi-regular polytopes".

[1]) The handbook quoted above contains in Vol II pages 256—261 some communications about the corresponding polytopes in the space Sp_n.

The following table shows the results which are obtained by substituting the values of e, k, f, r, p, q for the five different cases. For the sake of completeness those quantities are also included which indicate how many vertices are situated in face and bounding space. We must here notice that, the first cell having two kinds of faces and bounding bodies, we are obliged to take four new quantities, namely the numbers of vertices S_1 and T_1 in face and bounding body of one, the numbers of vertices S_2 and T_2 in face and bounding body of the second kind. Here S_2 and T_2 will relate to the truncating bodies with faces of the same kind and S_1 and T_1 refer to the truncated bodies, where we must then consider as far as S_1 is concerned those faces which the truncated bodies keep in common. We must likewise for the second cell, with two kinds of vertices and edges, introduce the four new quantities P_1', P_2', Q_1', Q_2',

As is evident $T_1 = Q_1' = k'$, whilst $T_2 = Q_2'$ is the number of vertices of the regular polyhedron with q faces.

	e	E, R'	K, F'	F, K'	R, E'	P, S'	Q, T'	S_1, P_1'	S_2, P_2'	T_1, Q_1'	T_2, Q_2'
C_5	5	10	30	30	10	3	5	3	3	6	4
C_8	16	32	96	88	24	3	5	4	3	12	4
C_{24}	24	96	288	240	48	3	5	4	4	12	8
C_{600}	120	720	3600	3600	720	3	7	3	3	6	12
C_{120}	600	1200	3600	3120	720	3	5	5	3	30	4

In a second part we shall submit each of these five pairs of new cells to a separate investigation.

Mathematics. — *"The analogon of the Cf. of* KUMMER *in seven-dimensional space"*. By Dr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of December 28, 1907).

§ 1. In a preceding communication a method was indicated to generate the *Cff.* in spaces of (2^p—1) dimensions, which can be regarded as analoga of the *Cf.* of KUMMER [1]).

[1]) A quotation in HUDSON's *Kummer's Quartic Surface* (p. 187) drew my attention to the fact, that these *Cff.* have already been obtained by an altogether different method by W. WIRTINGER *(Göttinger Nachrichten*, 1889, page 474; *Monatshefte fur Mathematik und Physik* I, page 113; *Mathem. Annalen* 40, page 74). In these papers the varieties are also investigated, for which the elements of such *Cff.* are singular in the same way as those of the *Cf.* (16_6) for KUMMER's quartic surface.