The following table shows the results which are obtained by substituting the values of $e, k, f, r, p, q$ for the five different cases. For the sake of completeness those quantities are also included which indicate how many vertices are situated in face and bounding space. We must here notice that, the first cell having two kinds of faces and bounding bodies, we are obliged to take four new quantities, namely the numbers of vertices $S_1$ and $T_1$ in face and bounding body of one, the numbers of vertices $S_2$ and $T_2$ in face and bounding body of the second kind. Here $S_2$ and $T_2$ will relate to the truncating bodies with faces of the same kind and $S_1$ and $T_1$ refer to the truncated bodies, where we must then consider as far as $S_1$ is concerned those faces which the truncated bodies keep in common. We must likewise for the second cell, with two kinds of vertices and edges, introduce the four new quantities $P_1'$, $P_2'$, $Q_1'$, $Q_2'$,

As is evident $T_1 = Q_1' = k'$, whilst $T_2 = Q_2'$ is the number of vertices of the regular polyhedron with $q$ faces.

| | $e$ | $E, R'$ | $K, F'$ | $F, K'$ | $R, E'$ | $P, S'$ | $Q, T'$ | $S_1, P_1'$ | $S_2, P_2'$ | $T_1, Q_1'$ | $T_2, Q_2'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_5$ | 5 | **10** | **30** | **30** | **10** | 3 | 5 | 3 | 3 | 6 | 4 |
| $C_8$ | 16 | **32** | **96** | **88** | **24** | 3 | 5 | 4 | 3 | 12 | 4 |
| $C_{24}$ | 24 | **96** | **288** | **240** | **48** | 3 | 5 | 4 | 4 | 12 | 8 |
| $C_{600}$ | 120 | **720** | **3600** | **3600** | **720** | 3 | 7 | 3 | 3 | 6 | 12 |
| $C_{120}$ | 600 | **1200** | **3600** | **3120** | **720** | 3 | 5 | 5 | 3 | 30 | 4 |

In a second part we shall submit each of these five pairs of new cells to a separate investigation.

**Mathematics.** — "*The analogon of the Cf. of* KUMMER *in seven-dimensional space*". By Dr. J. A. BARRAU. (Communicated by Prof. D. J. KORTEWEG).

(Communicated in the meeting of December 28, 1907).

§ 1. In a preceding communication a method was indicated to generate the *Cff.* in spaces of $(2^p—1)$ dimensions, which can be regarded as analoga of the *Cf.* of KUMMER [1]).

[1]) A quotation in HUDSON's *Kummer's Quartic Surface* (p. 187) drew my attention to the fact, that these *Cff.* have already been obtained by an altogether different method by W. WIRTINGER (*Göttinger Nachrichten*, 1889, page 474; *Monatshefte fur Mathematik und Physik* I, page 113; *Mathem. Annalen* 40, page 74). In these papers the varieties are also investigated, for which the elements of such *Cff.* are singular in the same way as those of the *Cf.* $(16_6)$ for KUMMER's quartic surface.

In the following the $K^{VII}$ is under closer investigation especially with a view to the *Cff.* obtainable out of it by omission of certain elements. To this end it is necessary to construct an other diagram than that of eight simplexes, which can only clearly show the *Cff.* $(56_{26})$; $(48_{22})$; $(40_{19})$; $(32_{16})$; $(24_{13})$ and $(16_{10})$, formed by omission of the elements of 1, 2, 3, 4, 5, 6 simplexes, all (except the first) in different types, likewise of *Cff.* $(24_9)$ and $(32_{12})$, constructed exclusively of fillings $(8_3)$.

§ 2. If we isolate in the *Cf.* of Kummer a point and a plane not incident to it, the remaining fifteen elements of each kind are divided into a sextuple incident to one of the isolated elements and a nonuple. Each one of the two sextuples forms with the 15 elements of the other kind a free *Cf.* $(6_5, 15_2)$ which means nothing else but that each of the fifteen right lines connecting the *Cf.*-points in one plane bears another *Cf*-plane and reciprocally.

The two nonuples of elements, however, of both kinds together form one *Cf.* $(9_4)$, the structure of which is identical to that of *Cf.* $(9_4)$ III of the classification of Martinetti [2]).

This arrangement can be done in $16 \times 10 = 160$ ways.

We can likewise isolate out of $K^{VII}$ in $64 \times 36 = 2304$ ways a point and an $Sp_6$ not incident to it, by which the sixty-three other elements of each kind are divided into a group of twenty-eight incident to the isolated element of the other kind and a remaining group of thirty-five. The two groups of twenty-eight form together a scheme $(28_{12})$; each group of twenty-eight with that of thirty-five of another kind a scheme $(28_{15}, 35_{12})$; addition of $(28_{12})$ and $(28_{15}, 35_{12})$ furnishes a scheme $(28_{27}, 63_{12})$; the two groups of thirty-five form together a scheme $(35_{16})$.

This arrangement made for the two elements $A1$ is shown in the plate, where the same notations are assigned to the elements as in the diagram of simplexes of which it is a transformation.

We have but to explain how the regular composition indicated by the thicker lines is obtained.

§ 3. Let us first take into consideration the scheme $(28_{27}, 63_{12})$ of points (columns) and $Sp_6$ (rows). Every number of twelve points on a row lying in two different *Cf.*-$Sp_6$ lies in an $Sp_5$, so we can take the *Cf.* to consist of twenty-eight points and sixty-three $Sp_5$ in $Sp_6$; each of the sixty-four *Cf.*-$Sp_6$ of the $K^{VII}$ contains such a *Cf.* (and reciprocally).

[2]) Atti della R. Accademia Peloritana XV.

We find for the complete notation of such a *Cf.* in a way analogous to that of § 6 of the former paper:

| | $Sp_0$ | $Sp_1$ | $Sp_2$ | $Sp_3$ | $Sp_4$ | $Sp_5$ |
|---|---|---|---|---|---|---|
| | **28** | **378** | **2016** | **5040** | **1008** | **63** |
| incid. to: | | | | | | |
| $Sp_0$ | — | 2 | 3 | 4 | 6 | 12 |
| $Sp_1$ | 27 | — | 3 | 6 | 15 | 66 |
| $Sp_2$ | 216 | 16 | — | 4 | 20 | 160 |
| $Sp_3$ | 720 | 80 | 10 | — | 15 | 240 |
| $Sp_4$ | 216 | 40 | 10 | 3 | — | 32 |
| $Sp_5$ | 27 | 11 | 5 | 3 | 2 | — |

By projection and intersection we find from this in $Sp_2$ a *Cf.* $(378_{16}, 2016_3)$; a $(2016_{10}, 5040_4)$ and a $(5040_3, 1008_{15})$ of points and right lines, in $Sp_3$ a $(378_{80}, 5040_6)$ and a $(2016_{10}, 1008_{20})$ of points and planes.

Although the number of *Cf*-points is $28 = 4 \times 7$ one cannot succeed in forming four simplexes $S_6$ out of the *Cf.*-elements; after isolation of such a simplex (which is possible in several ways) we can form out of the remaining elements (also in several ways) at most a scheme $S_5$, and then an $S_4$, $S_3$, $S_2$ and $S_1$ after which of the $(28_{12})$ an element of each kind is left, mutually not-incident; which we join to an "$S_0$". In the figure $S_3 + S_2 + S_1 + S_0$ are taken together to a scheme $(10_6)$ which we indicate by $T$.

The arrangement of the thirty-five remaining elements follows now by our regularly putting down the combinations 3 by 3 of the seven points chosen for $S_6$; it is evident that the entire diagram contains along the chief diagonal only schemes $S$ or $T$, whilst outside a couple of new fillings appear amongst which we notice a $(10_4)$, complementary to $T$.

It is by addition of these partial schemes that we can obtain a great number of *Cff.* included in the total figure; we restrict ourselves to the *forced Cff.* which are those of which each element shows more incidences than are sufficient to determine it and of which for this reason the existence is remarkable from a geometric point of view.

Of the *Cf.* $(28_{27}, 63_{12})$ in $Sp_6$ the twenty-eight points form evidently with twenty-eight $Sp_5$ a dual *Cf.* $(28_{12})$, the same points with the thirty-five other $Sp_5$ a *Cf.* $(28_{15}, 35_{12})$.

By omission out of $(28_{12})$ of a $S_6$ remains a $(21_{10})$; by omitting $S_6$ and $S_5$ a $(15_8)$ remains the scheme of which is *anallagmatic*: each couple of its $Cf.$-$Sp_5$ has four $Cf.$-points in common. The same number of 15 points forms with 15 other $Sp_5$ (namely $H1$ as far as $B1$ included out of the number of thirty-five) a $Cf.$ $(15_7)$ of which the scheme is complementary to the anallagmatic $(15_8)$.

Out of the $Cf.$ $(35_{16})$ is formed by omission of $S_4$ a $Cf.$ $(30_{14})$, by omission of $T$ a $Cf.$ $(25_{12})$, of $S_4$ and $T$ a $Cf.$ $(20_{10})$; of two different $T$ a $Cf.$ $(15_8)$, identical to the already mentioned one, its points lie in an $Sp_6$, the $(35_{16})$ has in each of the twenty-eight other $Cf.$-$R_6$ such a $(15_8)$.

If we add to the $Cf.$ $(30_{14})$ a system $T$ out of $(28_{12})$ a $Cf.$ $(40_{18})$ is formed.

The $Cf.$ $(35_{16})$ is also obtainable out of the diagram of simplexes of the $K^{VII}$, the simplex $A$ then falls away entirely, of each of the seven other ones three elements of each kind disappear. The diagram $(35_{16})$ consists of seven systems $S_4$ in the chief diagonal, mutually connected by fillings $(5_2)$, which all degenerate into $(3_2)$ and $(2_2)$.

By omitting 1, 2, 3, 4 from this $S_4$ we obtain $Cff$: $(30_{14})$, $(25_{12})$, $(20_{10})$ and $(15_8)$. The $(30_{14})$ is identical to the already mentioned one, the $(15_8)$ however is of another type. not anallagmatic, neither do its points lie in *one* $Sp_6$.

§ 5. In each of the $Sp_5$ formed by intersection of two $Cf.$-$Sp_6$ of $K^{VII}$ lie 12 $Cf.$-points, of which thirty-two sextuples are also common to a third $Cf.$-$Sp_6$; such a sextuple lies thus in an $Sp_4$, the twelve points form with the thirty-two $Sp_4$ a $Cf.(12_{16}, 32_6)$.

We can give to the diagram of such a $Cf.$ the following form· (see table p. 507).

If e.g. we take the $Cf$-$Sp_5$, formed by the intersection of the $Cf$-$Sp_6$. $A\,1$ and $A\,2$, the twelve points become respectively:

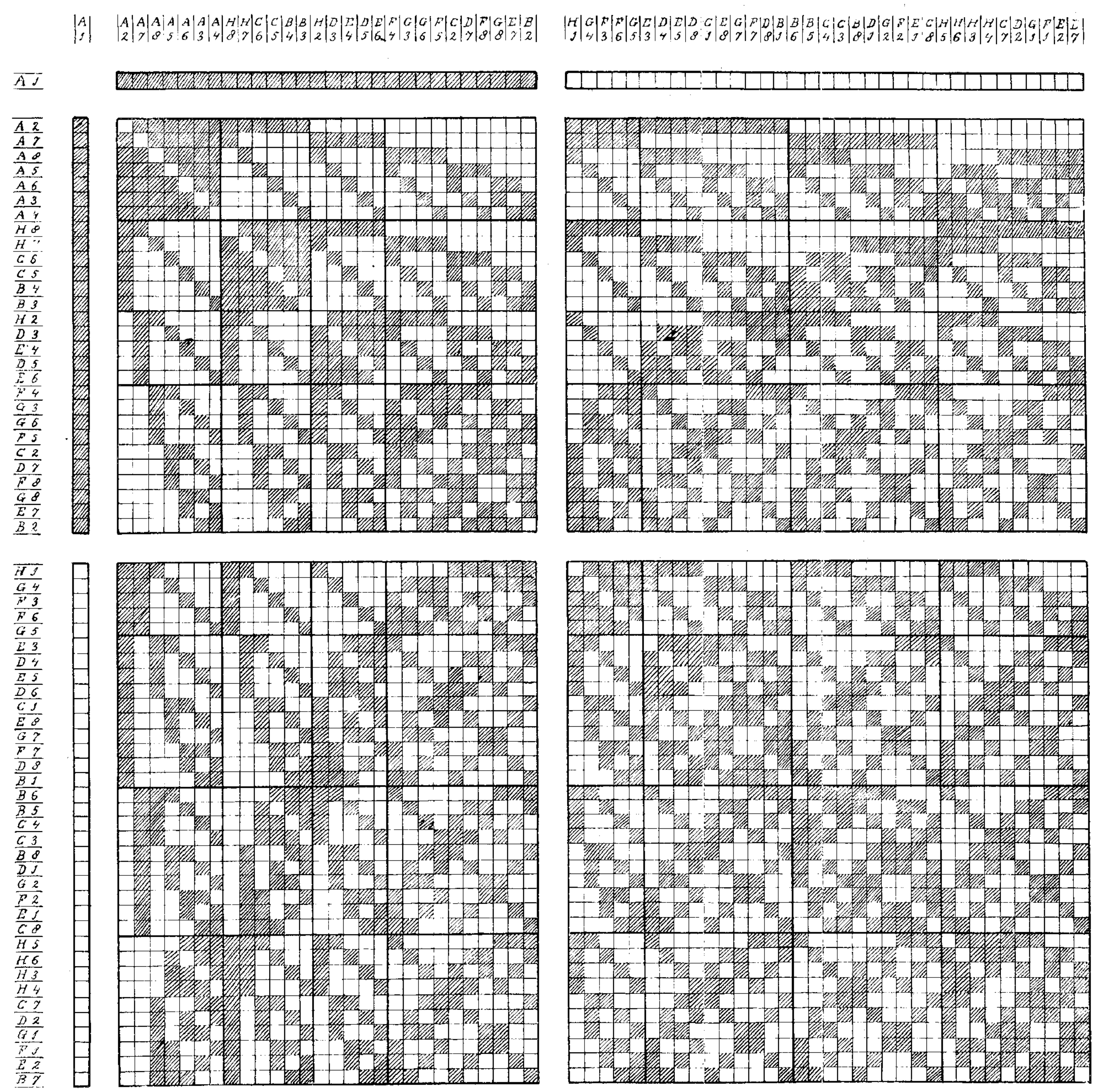
$$\begin{array}{ll} A\,3 = P\,1 & B\,4 = Q\,1 \\ A\,4 = P\,2 & B\,3 = Q\,2 \\ A\,5 = P\,3 & C\,6 = Q\,3 \\ A\,6 = P\,4 & C\,5 = Q\,4 \\ A\,7 = P\,5 & H\,8 = Q\,5 \\ A\,8 = P\,6 & H\,7 = Q\,6 \end{array}$$

The entire $Cf.$ consists evidently of two simplexes $S_5 : P$ and $Q$ in Mobius-position forming together the part $(12_6)$ whilst moreover every triplet of vertices of one simplex with the three non-conjugate ones of the other lie in *one* $Sp_4$, i.o.w., each face of one intersects the non-conjugate one of the other.

P Q

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 1 | | | | | |
| 1 | | 3 | 4 | 5 | 6 | | 2 | | | | |
| 1 | 2 | | 4 | 5 | 6 | | | 3 | | | |
| 1 | 2 | 3 | | 5 | 6 | | | | 4 | | |
| 1 | 2 | 3 | 4 | | 6 | | | | | 5 | |
| 1 | 2 | 3 | 4 | 5 | | | | | | | 6 |
| 1 | | | | | | | 2 | 3 | 4 | 5 | 6 |
| | 2 | | | | | 1 | | 3 | 4 | 5 | 6 |
| | | 3 | | | | 1 | 2 | | 4 | 5 | 6 |
| | | | 4 | | | 1 | 2 | 3 | | 5 | 6 |
| | | | | 5 | | 1 | 2 | 3 | 4 | | 6 |
| | | | | | 6 | 1 | 2 | 3 | 4 | 5 | |
| 1 | 2 | 3 | | | | | | | 4 | 5 | 6 |
| 1 | 2 | | 4 | | | | | 3 | | 5 | 6 |
| 1 | 2 | | | 5 | | | | 3 | 4 | | 6 |
| 1 | 2 | | | | 6 | | | 3 | 4 | 5 | |
| 1 | | 3 | 4 | | | | 2 | | | 5 | 6 |
| 1 | | 3 | | 5 | | | 2 | | 4 | | 6 |
| 1 | | 3 | | | 6 | | 2 | | 4 | 5 | |
| 1 | | | 4 | 5 | | | 2 | 3 | | | 6 |
| 1 | | | 4 | | 6 | | 2 | 3 | | 5 | |
| 1 | | | | 5 | 6 | | 2 | 3 | 4 | | |
| | 2 | 3 | 4 | | | 1 | | | | 5 | 6 |
| | 2 | 3 | | 5 | | 1 | | | 4 | | 6 |
| | 2 | 3 | | | 6 | 1 | | | 4 | 5 | |
| | 2 | | 4 | 5 | | 1 | | 3 | | | 6 |
| | 2 | | 4 | | 6 | 1 | | 3 | | 5 | |
| | 2 | | | 5 | 6 | 1 | | 3 | 4 | | |
| | | 3 | 4 | 5 | | 1 | 2 | | | | 6 |
| | | 3 | 4 | | 6 | 1 | 2 | | | 5 | |
| | | 3 | | 5 | 6 | 1 | 2 | | 4 | | |
| | | | 4 | 5 | 6 | 1 | 2 | 3 | | | |

J. A. BARRAU. "The analogon of the *Cf.* of Kummer in sevendimensional space."



This connection is for the first time possible in $Sp_5$, the analogon in $Sp_3$ would be: two tetrahedra in MÖBIUS-position, where each edge of one intersects the non-conjugate one of the other; of this *Cf.* $(8_7, 14_4)$, although it is possible to design the diagram, the execution is evidently impossible.

We find for the complete notation of *Cf.* $(12_{16}, 32_6)$:

| | $Sp_0$ | $Sp_1$ | $Sp_2$ | $Sp_3$ | $Sp_4$ |
|---|---|---|---|---|---|
| | **12** | **60** | **160** | **240** | **32** |
| incid. to | | | | | |
| $Sp_0$ | — | 2 | 3 | 4 | 6 |
| $Sp_1$ | 10 | — | 3 | 6 | 15 |
| $Sp_2$ | 40 | 8 | — | 4 | 20 |
| $Sp_3$ | 80 | 24 | 6 | — | 15 |
| $Sp_4$ | 16 | 8 | 4 | 2 | — |

By projecting and intersecting are formed out of these e.g. in $Sp_2$ a *Cf.* $(60_8, 160_3)$ and a *Cf.* $(160_6, 240_4)$ of points and right lines; in $Sp_3$ a *Cf.* $(60_{24}, 240_6)$ and a *Cf.* $(160_4, 32_{20})$ of points and planes.

§ 6. The points of $Sp_5$ can be conjugated one to one to the linear complexes of the usual three-dimensional space, the $Sp_4$ become linear systems of $\infty^4$ of these complexes, the *Cf.* $(12_{16}, 32_6)$ can be represented in our space.

It is however possible to take the twelve complexes simultaneously special and to regard them as right lines, the thirty-two $Sp_4$ then become linear complexes which each contain a sextuple of the right lines; the *Cf.* $(12_{16}, 32_6)$ is then realized in right lines and linear complexes of our space.

We can easily give line-coordinates for such a number of twelve right lines by omitting from the point-coordinates of the twelve points a couple, e. g. $X_7$ and $X_8$, and by letting the six remaining ones satisfy the fundamental relation

$$X_1 X_6 + X_2 X_5 + X_3 X_4 = 0$$

So we obtain e. g. the right lines

$$
\begin{array}{ll}
P1 = (c, -d, -a, \ 0, -g, \ h) & Q1 = (d, \ c, \ 0, -a, -h, -g) \\
P2 = (d, \ c, \ 0, -a, \ h, \ g) & Q2 = (c, -d, -a, \ 0, \ g, -h) \\
P3 = (0, -f, \ g, -h, -a, \ 0) & Q3 = (f, \ 0, \ h, \ g, \ 0, -a) \\
P4 = (f, \ 0, -h, -g, \ 0, -a) & Q4 = (0, -f, -g, \ h, -a, \ 0) \\
P5 = (g, -h, \ 0, \ f, \ c, -d) & Q4 = (h, \ g, -f, \ 0, \ d, \ c) \\
P6 = (h, \ g, \ f, \ 0, -d, -c) & Q6 = (g, -h, \ 0, -f, -c, \ d);
\end{array}
$$

if besides is satisfied

$$ch + dg = af - gh = 0.$$

The peculiarity appearing with this example taken for simplicity's sake, that the right lines show mutually some incidences, is lost by submitting the coordinates in $Sp_5$ first to a linear transformation.

In the same way, indeed, we can formulate for all *Cff.* indicated in spaces of a lower number of dimensions an analytical definition by deducing the coordinates of their elements from those of the elements of $K^{\mathrm{VII}}$.

**Chemistry:** — "*On the constitution of* VAN GEUNS's *oxymethyldinitrobenzonitrile*". By Dr. J. J. BLANKSMA. (Communicated by Prof. A. F. HOLLEMAN).

By the action of potassiumcyanide on meta-dinitrobenzene in methylalcoholic or ethylalcoholic solution, LOBRY DE BRUYN [1]) obtained in 1882 the oxymethyl- or oxyethylnitrobenzonitrile $C_6H_3(OCH_3)CN\,NO_2$ 1. 2. 3.

The investigation of these substances was continued afterwards by VAN GEUNS [2]) who succeeded in saponifying these nitriles to acidamines and in preparing the corresponding acids thereof. At the same time VAN GEUNS showed that in both substances a further nitro-group can be introduced by nitration with nitric and sulphuric acids thus yielding the compounds $C_6H_2(OCH_3)CN(NO_2)_2$ m.p. 113° and $C_6H_2(OC_2H_5)CN(NO_2)_2$ m.p. 63°, These two compounds contain a movable nitro-group which may be readily replaced by OH, $OCH_3$, $NH_2$, $NHCH_3$, $NHC_6H_5$ etc.

As, however, the place where the nitro-group had been introduced had remained unknown, the constitution of these derivatives was consequently also unknown.

When VAN GEUNS, owing to his departure for India was obliged

1) Recueil 2, 205.

2) Dissertation Amsterdam 1903.