Mathematics. — *"Fourdimensional nets and their sections by spaces"*. (First part). By Prof. P. H. SCHOUTE.

(Communicated in the meeting of January 25, 1908).

Out of the table

$$C_5 \ldots . 75^\circ\, 31'\, 21'', \quad C_{16} \ldots . 120^\circ, \quad C_{120} \ldots . 144^\circ$$
$$C_8 \ldots . 90^\circ \qquad\quad , \quad C_{24} \ldots . 120^\circ, \quad C_{600} \ldots . 164^\circ\, 28'\, 39''$$

of the angles formed by two bounding bodies meeting in a face of the regular cells of space Sp_4 it is immediately evident that only for the cells C_8, C_{16}, C_{24} can there be any question about each respectively filling that space. It is well known, that this is really the case. In the handbook included in the Sammlung SCHUBERT *"Mehrdimensionale Geometrie"* (vol. II, page 241) is indicated how the two nets of the cells C_{16} and C_{24} can be deduced by transformation from the net of cells C_8, the existence of which is clear in itself. We repeat this here in a somewhat different form to add new considerations to it.

1. The points with the coordinates $(\pm 1, \pm 1, \pm 1, \pm 1)$ are the vertices of an eightcell $C_8^{(2)}$ with double the unit of length as length of edge, the origin of the coordinates as centre and the directions of the axes as directions of the edges. These vertices can be easily arranged in two groups of eight points, one group of which contains the points with a positive product of coordinates, the other group the points with a negative one. Each of these groups has the property that no two of the eight points are united by an edge of $C_8^{(2)}$; therefore we call them groups of non-adjacent vertices. Let us join for each of these groups the two points lying in the same face of $C_8^{(2)}$ by a diagonal, then the systems of edges of two cells $C_{16}^{(2\sqrt{2})}$ are generated; as each of the bounding cubes of $C_8^{(2)}$ is circumscribed about one of the 16 bounding tetrahedra of each of the two $C_{16}^{(2\sqrt{2})}$, we call these last inscribed in $C_8^{(2)}$, where one may be called positive, the other negative.

Let us now suppose the net of the C_8 to be composed of alternate white and black eightcells, so that two C_8 with a common bounding body differ in colour — from which it follows, that two C_8 in contact of edges do this too, whilst on the other hand two C_8 in face or in vertex contact bear the same colour —, and let us assume that in each white C_8 is inscribed a positive C_{16} and in each black C_8 a negative one; then it is clear that both groups of C_{16} do not yet fill the whole space Sp_4. For to make of a C_8 the inscribed C_{16} we must truncate from this measure polytope at each of the eight vanishing vertices a fivecell rectangular at this point, of which the four edges passing through this point have a length 2. Because a vertex which vanishes for one of the sixteen cells C_8, to which it belongs, does this for all, there will remain round this point sixteen alternate white and black rectangular fivecells and these will form together a $C_{16}^{(2\sqrt{2})}$ of which this point is the centre. Thus a space-filling for Sp_4 is formed by three equally numerous groups of cells $C_{16}^{(2\sqrt{2})}$ with the property that all cells C_{16} of the same group can be made to cover one another by translation.

To show how striking the regularity of the net of the C_{16} is we must suppose three cells $C_{16}^{(2\sqrt{2})}$, of which no two belong to the same group, to be removed parallel to themselves to a common centre, the origin of coordinates. We then see immediately that the vertices of the three $C_{16}^{(2\sqrt{2})}$ together form the vertices of a $C_{24}^{(2)}$. For the two inscribed cells $C_{16}^{(2\sqrt{2})}$ together again furnish the vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$ of the original eightcell $C_8^{(2)}$ and the coordinates of the vertices of the third cell $C_{16}^{(2\sqrt{2})}$ are

$$(\pm 2, 0, 0, 0), \ (0, \pm 2, 0, 0), \ (0, 0, \pm 2, 0), \ (0, 0, 0, \pm 2),$$

from which is evident what was assumed (compare "*Mehrdimensionale Geometrie*", vol II, p. 205).

We shall presently use this observation to trace the connection between the four groups of axes of the three systems of cells C_{16} with the groups of axes of C_8.

2. To transform the net of the cells C_8 into a net of cells C_{24} we must again suppose the cells of the former alternately coloured white and black in order to break up each of the black cells into eight congruent pyramids with the centre of the eightcell as common vertex and the eight bounding cubes as bases. By adding to each white eightcell the eight black pyramids having a bounding cube in common with it, the net of the cells $C_{24}^{(2)}$ is generated; in reality to the sixteen vertices of the eightcell supposed to be white with the origin of coordinates as centre, viz. to the points $(\pm 1, \pm 1, \pm 1, \pm 1)$ the eight vertices mentioned above

$$(\pm 2, 0, 0, 0), \ (0, \pm 2, 0, 0), \ (0, 0, \pm 2, 0), \ (0, 0, 0, \pm 2)$$

are added.

The transformation of the net of the $C_8^{(2)}$ into that of C_{24} can also take place in the following simple way. Divide each of the cells $C_8^{(2)}$

into 16 equal and similarly placed cells $C_8^{(1)}$ by means of four spaces through the centre O parallel to the pairs of bounding spaces. Then divide each of the sixteen parts $C_8^{(1)}$ (fig. 1) by the space in the midpoint of the diagonal concurring in the centre O of $C_8^{(2)}$ normal to this line into two equal halves; here the section as is known is a regular octahedron $A_{12} A_{13} \ldots A_{34}$. We now direct our attention first to the half cells $C_8^{(1)}$ surrounding the point O; they form a $C_{24}^{(\sqrt{2})}$. Of the 24 bounding octahedra sixteen are furnished by the sections $A_{12} A_{13} \ldots A_{34}$, whilst the eight remaining ones are obtained by joining

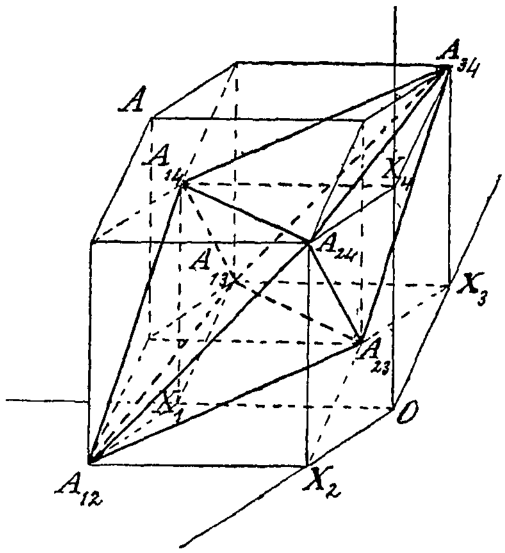

Fig. 1.

in each of the eight ends of the chords along the four axes OX_1, OX_2, OX_3, OX_4 through O, e. g. in X_1, the eight rectangular tetrahedra $X_1 (A_{12} A_{13} A_{14})$, where it is clear that in X_1 eight of those tetraheda really meet, because we can reverse the direction of each of the segments X_1A_{12}, X_1A_{13}, X_1A_{14}. Furthermore we observe that around an arbitrary vertex A of the original cell also 16 half cells $C_8^{(1)}$ are lying and that these form in exactly the same way a $C_{24}^{(\sqrt{2})}$. By this the net of the $C_8^{(2)}$ has been transformed into a net of cells $C_{24}^{(\sqrt{2})}$, where the centres and the vertices of the cells $C_8^{(2)}$ form the centres of the cells $C_{24}^{(\sqrt{2})}$ placed in the same way.

If we add to the considered sixteenth part $C_8^{(1)}$ (fig. 1) the three parts generated by reversing the sign of one of the two axes OX_1 and OX_2 or of both, it is immediately evident that A_{34} is the centre of a face of the original cell $C_8^{(2)}$. From this is evident to the eye the truth of the wellknown theorem, that the centres of the faces of a $C_8^{(2)}$ — and therefore also the centres of the edges of each of the two inscribed cells $C_{16}^{(2\sqrt{2})}$ — are the vertices of a $C_{24}^{(\sqrt{2})}$.

3. Before examining more closely the nets of the cells C_8, C_{16}, C_{24} — or, as we shall express ourselves, the nets (C_8), (C_{16}), (C_{24}) — in their mutual connection we put to ourselves the question whether it is possible to fill Sp_4 entirely with *different* regular cells. Here the table given above points to two possibilities. We can either complete the sum of the angles $75° 31' 21'$ and $164° 28' 39''$ with $120°$ to $360°$ or by combination of one of the two cells C_{16}, C_{24} with twice the other arrive at $360°$. The latter is however already excluded by the fact that C_{16} and C_{24} differ in bounding bodies, which obstacle does not occur when one tries to arrange the three cells C_5, C_{16}, C_{600} with the same length of edges around a face. Yet, though this is possible, neither in this way does one arrive at the object in view. If the indicated space-filling had taken place then two bounding tetrahedra of C_5, having always a face in common, would have to differ from each other in this, that one would at the same time have to belong to a C_{16} and the other to a C_{600} and this is impossible. For one cannot colour the bounding tetrahedra of a C_5 alternately white and black for the mere reason, that the number five of those tetrahedra is odd. So there is no space-filling of Sp_4 where *different* regular cells appear.

4. We shall now consider more closely the systems of points formed by the centres of the regular cells of the nets (C_8), (C_{16}), (C_{24}) which we shall indicate by the symbols (P_8), (P_{16}), (P_{24}).

Of the systems of points (P_8), (P_{16}), (P_{24}),, which we might call fourdimensional "assemblages of Bravais", (P_8) is the simplest. If the axes of coordinates are assumed through the centre of a definite cell $C_8^{(2)}$ parallel to the edges of this cell, then (P_8) is the system of the points $(2a_1, 2a_2, 2a_3, 2a_4)$ with only even integer coordinates which we indicate by means of abbreviated symbols by the equation $(P_8) = (2a_i)$.

Of the two other systems of points, (P_{24}) can be most simply expressed in (P_8). Out of the second mode of transformation of the cells $C_8^{(2)}$ into the cells $C_{24}^{(\sqrt{2})}$ it was clear to us that (P_{24}) is found by joining the system (P_8) to the system of the vertices of the cells $C_8^{(2)}$. Now this system of the vertices can be deduced out of (P_8) by a translation indicated in direction and magnitude by the line-segment connecting the centre of the eightcell, which served to determine the system of coordinates, with one of the vertices; thus this system of vertices is indicated in the same symbols by $(2a_i + 1)$ and we find $(P_{24}) = (2a_i) + (2a_i + 1)$, i. e. (P_{24}) is the system of the points with integer coordinates which are either all even or all odd.

Finally (P_{16}) is derived from (P_{24}) by adding to (P_8) not the whole system of the vertices of the cells $C_8^{(2)}$, but only that half which is not occupied by the vertices of the inscribed $C_{16}^{(2\sqrt{2})}$. We express this by means of the equation $P_{16} = (2a_i) + \frac{1}{2}(2a_i + 1)$.

Here we have to understand by $\frac{1}{2}(2a_i + 1)$ that system of points of which the coordinates are only odd integer numbers under the condition that half the sum is either always even or always odd. If in the cell $C_8^{(2)}$ which furnished us above with the system of coordinates a positive $C_{16}^{(2\sqrt{2})}$ is inscribed, which for the future we shall always suppose, then the point $(1, 1, 1, 1)$ is occupied by a vertex of the inscribed $C_{16}^{(2\sqrt{2})}$ and so for the non-occupied vertices $\frac{1}{2}(2a_i+1)$ half the sum of the four quantities a_i is odd.

If we make the connection between the systems of points (P_8), (P_{16}), (P_{24}) in the indicated way, then the number of points of (P_{24}) is twice, and the number of points (P_{16}) is one and a half times as large as that of (P_8) and so the fourdimensional volumes of $C_8^{(2)}$, $C_{16}^{(2\sqrt{2})}$, $C_{24}^{(\sqrt{2})}$ have to be in the same ratio as the numbers $1, \frac{2}{3}, \frac{1}{2}$. This can be easily verified. To make a $C_{16}^{(2\sqrt{2})}$ of $C_8^{(2)}$ we have truncated at eight vertices a rectangular fivecell, which is $\frac{1}{24}$ of $C_8^{(2)}$; so $\frac{2}{3}$ of $C_8^{(2)}$ remains. And to make of $C_8^{(2)}$ the cell $C_{24}^{(\sqrt{2})}$ contained in the former we have halved each of the sixteen parts $C_8^{(1)}$.

5. By the "transformation-view" of each of the nets (C_8), (C_{16}) and (C_{24}) with respect to a space Sp_3 of the bearing space Sp_4 as screen we understand the intersection varying every moment, of this non-moving space with the fourdimensional net moving along in the direction normal to this space. If for this movement we interchange the relative and the absolute, we can also take this transformation-view to be generated by the intersection of the non-moving fourdimensional net with a space Sp_3, moving along in a perpendicular direction and remaining parallel to itself; there we can again assume that this view is observed by one who shares the movement of the space

Sp_3. The chief aim of this communication is to indicate how we can connect the transformation-views of the nets (C_{16}), (C_{24}) with that of the net (C_8), which is by far the simplest. Because the three views furnish at every moment a filling of the intersecting space, this investigation can lead to new threedimensional space-fillings, even though they be not entirely regular.

To be able to design a transformation-view of the net (C_{16}) we must know for each of the component cells C_{16} the place *of* the centre and the position *about* the centre; as the coordinates of the centres of the cells are given above, we have only to occupy ourselves further with the position about the centre. We designate that position by means of the four diagonals of each C_{16} and we then notice that these four lines for each of the two kinds of inscribed cells C_{16} are also diagonals — groups of non-adjacent diagonals — of the circumscribed cells C_8, whilst for the cells C_{16} of the third group they are parallel to the axes of coordinates.

If we suppose the centre of a cell $C_{16}^{(2\sqrt{2})}$ of the third group to be at the same time the centre of a cell $C_8^{(4)}$, the edges of which are parallel to the axes of coordinates, the $C_{16}^{(2\sqrt{2})}$ is inscribed in this new eightcell in such a sense, that the vertices of $C_{16}^{(2\sqrt{2})}$ are the centres of the eight bounding cubes of $C_8^{(4)}$. For an obvious reason we call this $C_{16}^{(\sqrt{2})}$ *polarly* inscribed in $C_8^{(4)}$ — and now to distinguish, we call the cells of the two other groups *bodily* inscribed in the cells $C_8^{(2)}$. For, as was observed above, in each of the eight bounding cubes of $C_8^{(2)}$ a bounding tetrahedron of $C_{16}^{(2\sqrt{2})}$ is inscribed, whilst each of the remaining eight bounding tetrahedra of $C_{16}^{(2\sqrt{2})}$ has with respect to each of the four pairs of opposite bounding cubes of $C_8^{(2)}$ three vertices of one and one vertex of the other cube as vertices.

In this way each of the cells $C_{16}^{(2\sqrt{2})}$ of the net (C_{16}) is packed up in a C_8 as small as possible, of which the edges are parallel to the axes of coordinates; here the fourdimensional *cases* of the "erect" cells C_{16} of the third group are cells $C_8^{(4)}$, those of the "inclining" cells C_{16} of the first and the second group are cells $C_8^{(2)}$. Whilst the cases $C_8^{(2)}$ of the inclining cells C_{16} fill the space Sp_4, the cases $C_8^{(4)}$ of the erect cells C_{16} do so eight times, because $C_{16}^{(2\sqrt{2})}$ is the $\frac{1}{24}^{\text{th}}$ part of $C_8^{(4)}$, — as is immediately evident when one divides the erect $C_{16}^{(2\sqrt{2})}$ and its case $C_8^{(4)}$ by spaces through the common centre parallel to the pairs of bounding spaces of $C_8^{(4)}$ into sixteen equal parts and when one compares the rectangular fivecell of $C_{16}^{(2\sqrt{2})}$ to the $C_8^{(2)}$ of $C_8^{(4)}$ —, and the erect C_{16} together fill a third of Sp_4.

In the second mode of transformation of the cells $C_8^{(2)}$ of the net (C_8) into the cells $C_{24}^{(\sqrt{2})}$ of a net (C_{24}) the vertices of the $C_{24}^{(\sqrt{2})}$ concentric to $C_8^{(2)}$ are the centres of the faces of these $C_8^{(2)}$, from which it follows that the six centres of the faces of each of the eight bounding cubes of $C_8^{(2)}$ are vertices of a bounding octahedron of $C_{24}^{(\sqrt{2})}$ and so this cell may again be called inscribed — and *bodily* inscribed too — in $C_8^{(2)}$. Also the remaining bounding octahedra can be directly indicated with respect to these circumscribed $C_8^{(2)}$; through each of the sixteen vertices of $C_8^{(2)}$ pass six faces of this cell, of which the centres form the vertices of a bounding octahedron of $C_{24}^{(\sqrt{2})}$. [1])

From the preceding it follows, that the fourdimensional cases, inclosing the cells $C_{24}^{(\sqrt{2})}$ and having edges parallel to the axes of coordinates, consist of two nets (C_8) of cells $C_8^{(2)}$, which by exchange of centres and vertices pass into each other.

6. We conclude this first part by indicating the connection existing between the systems of axes of the five different cells with the origin of coordinates as common centre, which can be obtained by parallel translation of one of the cells $C_8^{(2)}$, one of each of the three groups of cells $C_{16}^{(2\sqrt{2})}$ and one of the cells $C_{24}^{(\sqrt{2})}$. We indicate these cells for brevity by C_8, C_{16}, C'_{16}, C''_{16}, C_{24} where C_{16} represents the polarly inscribed sixteencell and C'_{16} and C''_{16} successively the positive and the negative bodily inscribed one. Further here too — according to the notation of the handbook mentioned above — E, K, F, R will denote a vertex, midpoint of edge, centre of face, centre of bounding body and therefore OE, OK, OF, OR will have to denote the axes converging in these points. Thus OE_8 is an axis OE of C_8, OK_{16} an axis OK of C_{16}, OF'_{16} an axis OF of C'_{16}, etc.

The numbers of axes OE, OK, OF, OR of each of the three different cells are always the halves of the numbers of the elements E, K, F, R; they are contained in the following table.

Here C_{16} of course represents the three cells C_{16}, C'_{16}, C''_{16}.

We now indicate the connection of the systems of axes of the

[1]) By doubling the radii vectores of the six centres of the faces from the chosen vertex of these $C_8^{(2)}$ we find the central section normal to the diagonal of this point.

	OE	OK	OF	OR
C_8	8	16	12	4
C_{16}	4	12	16	8
C_{24}	12	48	48	12

five cells C_8, C_{16}, C'_{16}, C''_{16}, C_{24} by giving the coordinates of the points E, K, F, R belonging to these concentric cells with respect to two systems of axes of coordinates with the common centre of the cells as origin, the systems (OX_i) of the four axes OR_8 and the system (OY_i) of the four axes OE'_{16} (fig. 2) between which the relations

$$\left.\begin{array}{rl} 2y_1 = & x_1 + x_2 + x_3 + x_4 \\ 2y_2 = & x_1 - x_2 - x_3 + x_4 \\ 2y_3 = & -x_1 + x_2 - x_3 + x_4 \\ 2y_4 = & -x_1 - x_2 + x_3 + x_4 \end{array}\right\}$$

exist. [1])

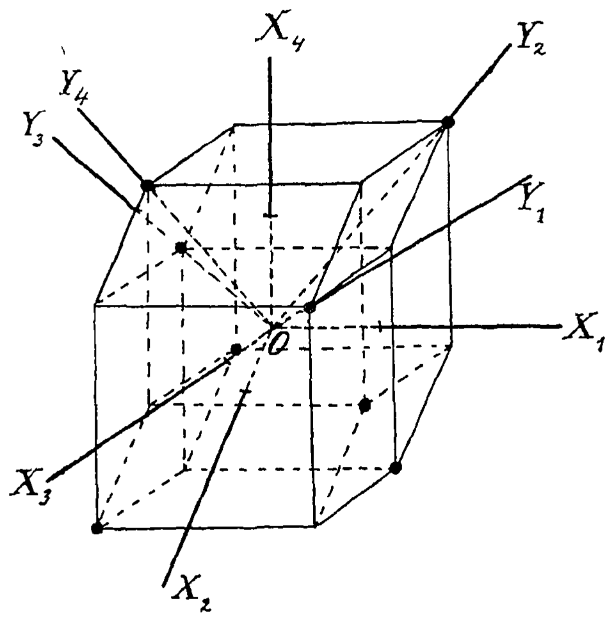

Fig. 2.

[1]) We selected this transformation T, because it causes the octuples of vertices of C_{16} and C'_{16} to pass into each other and those of C''_{16} into itself. It satisfies the condition $T^4 = -1$, so that first T^8 gives unity. We find that T^2 is a rectangular double-rotation round O by which (x_1, x_4) passes into $(-x_4, x_1)$ and (x_2, x_3) into $(-x_3, x_2)$.

We shall now give in both systems of coordinates the coordinates of the vertices of the five concentric cells and we divide in doing so — see the following table — the sixteen vertices of $C_8^{(2)}$ into the eight vertices of C'_{16} and the eight vertices of C''_{16}; to that end it is necessary for distinction to indicate whether the product of the coordinates is positive on negative.

Cells	Number of vertices	Coordinates (OX_i)	Product	Coordinates (OY_i)	Product
C_8 and C'_{16}	8	$(\pm 1, \pm 1, \pm 1, \pm 1)$	+	$(\pm 2, 0, 0, 0)$	
C_8 and C''_{16}	8	$(\pm 1, \pm 1, \pm 1, \pm 1)$	—	$(\pm 1, \pm 1, \pm 1, \pm 1)$	—
C_{16}	8	$(\pm 2, 0, 0, 0)$		$(\pm 1, \pm 1, \pm 1, \pm 1)$	+
C_{24}	24	$(\pm 1, \pm 1, 0, 0)$		$(\pm 1, \pm 1, 0, 0)$	

With the aid of this it is easy to find both quadruples of coordinates of the systems of the points K, F, R of the five cells. They are given in the following table, which after all the preceding is clear in itself.

Cells					Number of axes	Coordinates (OX_i)	Product	Coordinates (OY_i)	Product
C_8	C_{16}	C'_{16}	C''_{16}	C_{24}					
E	$2R$	E	$\frac{4}{3}R$	$2R$	4	$(\pm 1, \pm 1, \pm 1, \pm 1)$	+	$(2, 0, 0, 0)$	
E	$2R$	$\frac{4}{3}R$	E	$2R$	4	$(\pm 1, \pm 1, \pm 1, \pm 1)$	—	$(\pm 1, \pm 1, \pm 1, \pm 1)$	—
K	$\frac{3}{2}F$	—	—	$\frac{3}{2}F$	16	$(\pm 1, \pm 1, \pm 1, 0)$		$\left(\pm\frac{3}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$	—
F	K	K	K	E	12	$(\pm 1, \pm 1, 0, 0)$		$(\pm 1, \pm 1, 0, 0)$	
R	$\frac{1}{2}E$	R	R	R	4	$(\pm 1, 0, 0, 0)$		$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$	+
—	—	F	—	F	16	$\left(\pm 1, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$	—	$\left(\pm\frac{2}{3}, \pm\frac{2}{3}, \pm\frac{2}{3}, 0\right)$	
—	—	—	F	F	16	$\left(\pm 1, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$	+	$\left(\pm 1, \pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$	+
—	—	—	—	K	48	$\left(\pm 1, \pm\frac{1}{2}, \pm\frac{1}{2}, 0\right)$		$\left(\pm 1, \pm\frac{1}{2}, \pm\frac{1}{2}, 0\right)$	

Of course the axes, of which the number is given each time, agree in nature with the points connected by them with O. So the four axes given in the first row are axes OE for C_8 and C'_{16}, axes OR for C_{16}, C'''_{16} and C_{24}; moreover the coefficients 2, $\frac{4}{3}$, 2 of $2R$, $\frac{4}{3}R$, $2R$ indicate that the quadruples of coordinates appearing in this row relate to the point which is obtained by multiplying the observed axis OR of C_{16}, C'''_{16}, C_{24} as far as the length from O goes by 2, $\frac{4}{3}$, 2.

With the preceding we have pointed out the position of each axis of one of the cells of the three nets (C_8), (C_{16}), (C_{24}) with reference to each of the two systems of coordinates and so we have furnished in connection with the preceding the material by which it is possible to deduce easily all the spacial sections of these three regular nets connected in a simple way with these axes. To give an example here already we observe that a space normal to one of the twelve axes OF_8 is normal to an axis OK for all the cells of the net (C_{16}); if it now proves possible to determine such a space in such a way that it is equally distant from the centres of all the cells C_{16} which are intersected, then in the intersecting space a more or less regular space-filling is generated by a selfsame body in three different positions.

In a future part we hope to commence with the determination of the remarkable spacial sections of the nets (C_8), (C_{16}), (C_{24}).

Mathematics. — "*Contribution to the knowledge of the surfaces with constant mean curvature*". By Dr. Z. P. Bouman. (Communicated by Prof. Jan de Vries).

(Communicated in the meeting of January 25, 1908).

§ 1. As is known the great difficulty connected with the study of the surfaces with constant mean curvature is the integration of the differential equation

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = -\sinh \theta \,.\, \cosh \theta.$$

The course followed here leads to two simultaneous partial differential equations of order one and of degree two.

In Gauss' symbols the value of the mean curvature H of a surface is indicated by

$$H = \frac{2FD' - ED'' - GD}{EG - F^2}.$$