four axes given in the first row are axes OE for C_8 and C'_{16}, axes OR for C_{16}, C'''_{16} and C_{24}; moreover the coefficients $2, \frac{4}{3}, 2$ of $2R, \frac{4}{3}R, 2R$ indicate that the quadruples of coordinates appearing in this row relate to the point which is obtained by multiplying the observed axis OR of C_{16}, C'''_{16}, C_{24} as far as the length from O goes by $2, \frac{4}{3}, 2$.

With the preceding we have pointed out the position of each axis of one of the cells of the three nets (C_8), (C_{16}), (C_{24}) with reference to each of the two systems of coordinates and so we have furnished in connection with the preceding the material by which it is possible to deduce easily all the spacial sections of these three regular nets connected in a simple way with these axes. To give an example here already we observe that a space normal to one of the twelve axes OF_8 is normal to an axis OK for all the cells of the net (C_{16}); if it now proves possible to determine such a space in such a way that it is equally distant from the centres of all the cells C_{16} which are intersected, then in the intersecting space a more or less regular space-filling is generated by a selfsame body in three different positions.

In a future part we hope to commence with the determination of the remarkable spacial sections of the nets (C_8), (C_{16}), (C_{24}).

Mathematics. — "*Contribution to the knowledge of the surfaces with constant mean curvature*". By Dr. Z. P. BOUMAN. (Communicated by Prof. JAN DE VRIES).

(Communicated in the meeting of January 25, 1908).

§ 1. As is known the great difficulty connected with the study of the surfaces with constant mean curvature is the integration of the differential equation

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = - \sinh\theta \, . \cosh\theta .$$

The course followed here leads to two simultaneous partial differential equations of order one and of degree two.

In Gauss' symbols the value of the mean curvature H of a surface is indicated by

$$H = \frac{2FD' - ED'' - GD}{EG - F^2} .$$

As independent coordinates on the surface we choose those which are invariable along the lines with length zero and we represent them by ξ and η. So we find.

$$H = -2\frac{D'}{F}, \quad \text{whilst } E = G = 0\,.$$

Let us multiply both members of the first equation by X (cosine of the angle of the normal with the X-axis); we then find:

$$FHX = -2D'X\,.$$

But

$$D'X = \frac{d^2x}{\partial\xi\partial\eta} \text{ [1]}$$

and moreover [2]:

$$FX = \frac{1}{i}\begin{vmatrix} \frac{\partial y}{\partial \xi} & \frac{dz}{\partial \xi} \\ \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{vmatrix} = \frac{1}{i}\begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix},$$

where x, y, z represent the Cartesian coordinates of the surface with respect to a rectangular system of axes.

So we find

$$\frac{H}{i}\begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix} = -2\frac{\partial^2 x}{\partial\xi\partial\eta},$$

or:

and likewise:

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial\xi\partial\eta} &= -\frac{H}{2i}\begin{pmatrix} y & z \\ \xi & \eta \end{pmatrix} \\ \frac{\partial^2 y}{\partial\xi\partial\eta} &= -\frac{H}{2i}\begin{pmatrix} z & x \\ \xi & \eta \end{pmatrix} \\ \frac{\partial^2 z}{\partial\xi\partial\eta} &= -\frac{H}{2i}\begin{pmatrix} x & y \\ \xi & \eta \end{pmatrix} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (I)$$

Moreover x, y and z must satisfy

$$E = G = 0,$$

therefore

$$\left.\begin{aligned} \Sigma\left(\frac{\partial x}{\partial \xi}\right)^2 &= 0 \\ \Sigma\left(\frac{\partial x}{\partial \eta}\right)^2 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (II)$$

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[1]) BIANCHI, Vorlesungen über Differential-Geometrie, translation into German by MAX LUKAT, page 89.

[2]) l. c. page 86.

The equations (I) and (II) give back for $H=0$ the problem of the minimal surfaces.

For $-\frac{H}{2i}$ we shall introduce for brevity the symbol Q.

§ 2. To satisfy beforehand (II) we put

$$\left.\begin{array}{ll}\frac{\partial x}{\partial \xi}+i\frac{\partial y}{\partial \xi}=u\frac{\partial z}{\partial \xi}\,, & \frac{\partial x}{\partial \eta}+i\frac{\partial y}{\partial \eta}=v\frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \xi}-i\frac{\partial y}{\partial \xi}=-\frac{1}{u}\frac{\partial z}{\partial \xi}\,, & \frac{\partial x}{\partial \eta}-i\frac{\partial y}{\partial \eta}=-\frac{1}{v}\frac{\partial z}{\partial \eta}\end{array}\right\}, \quad . \quad . \quad (III)$$

where u and v are functions to be determined of ξ and η.

When we substitute the equations (III) into (I) we find the equations which u and v must satisfy, whilst moreover $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$, derived from (III) must obey the conditions of integrability.

The latter furnish

$$\frac{\partial\, u\frac{\partial z}{\partial \xi}}{\partial \eta}=\frac{\partial\, v\frac{\partial z}{\partial \eta}}{\partial \xi},$$

and

$$\frac{\partial\, \frac{1}{u}\frac{\partial z}{\partial \xi}}{\partial \eta}=\frac{\partial\, \frac{1}{v}\frac{\partial z}{\partial \eta}}{\partial \xi},$$

which is clear.

Writing out we find

$$\left.\begin{array}{ll}(a). & \frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}+u\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}=\frac{\partial v}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}+v\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta} \\ (b). & \frac{1}{u^2}\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}-\frac{1}{u}\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}=\frac{1}{v^2}\frac{\partial v}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}-\frac{1}{v}\frac{\partial^2 z}{\partial \xi\,.\,\partial \eta}\end{array}\right\} \quad . \quad . \quad (IV)$$

If we now also substitute the values of $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$ into the equations (I) whilst we put $Q=-\frac{H}{2i}$ we find:

$$\left(\frac{\partial u}{\partial \eta}+\frac{1}{u^2}\frac{\partial u}{\partial \eta}\right)\frac{\partial z}{\partial \xi}+\left(u-\frac{1}{u}\right)\frac{\partial^2 z}{\partial \xi \partial \eta}=\frac{Q}{i}\left(u+\frac{1}{u}-v-\frac{1}{v}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta},$$

$$\frac{1}{i}\left(\frac{\partial u}{\partial \eta}-\frac{1}{u^2}\frac{\partial u}{\partial \eta}\right)\frac{\partial z}{\partial \xi}+\frac{1}{i}\left(u+\frac{1}{u}\right)\frac{\partial^2 z}{\partial \xi \partial \eta}=Q\left(v-\frac{1}{v}-u+\frac{1}{u}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta},$$

$$\frac{\partial^2 z}{\partial \xi \partial \eta}=\frac{Q}{2i}\left(\frac{u}{v}-\frac{v}{u}\right)\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}.$$

From these three last equations we derive directly with the aid of (IV):

$$\left.\begin{array}{ll}(a). & \dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial z}{\partial\xi}+u\,\dfrac{\partial^2 z}{\partial\xi\partial\eta}=Q\,i\,(v-u)\,\dfrac{\partial z}{\partial\xi}\cdot\dfrac{\partial z}{\partial\eta}\\ (b). & \dfrac{1}{u^2}\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial z}{\partial\xi}-\dfrac{1}{u}\dfrac{\partial^2 z}{\partial\xi\partial\eta}=Q\,i\left(\dfrac{1}{v}-\dfrac{1}{u}\right)\dfrac{\partial z}{\partial\xi}\cdot\dfrac{\partial z}{\partial\eta}\\ \text{whilst} & \dfrac{\partial^2 z}{\partial\xi\partial\eta}.=\dfrac{Q}{2i}\left(\dfrac{u}{v}-\dfrac{v}{u}\right)\dfrac{\partial z}{\partial\xi}\cdot\dfrac{\partial z}{\partial\eta}\end{array}\right\}\quad\ldots\quad(V)$$

We can easily show that one of the equations (V) is dependent on the two others, as is clear.

If we divide both members of (V,a) by u^2 and if we add (V,b), we find:

$$\frac{\partial z}{\partial\eta}=\frac{2v}{Qi(v-u)^2}\cdot\frac{\partial u}{\partial\eta}.$$

From (IV,a) follows:

$$\frac{\partial u}{\partial\eta}\cdot\frac{\partial z}{\partial\xi}-\frac{\partial v}{\partial\xi}\cdot\frac{\partial z}{\partial\eta}=(v-u)\frac{\partial^2 z}{\partial\xi\partial\eta}=(v-u)\left(\frac{u^2-v^2}{uv}\right)\frac{Q}{2i}\cdot\frac{\partial z}{\partial\xi}\cdot\frac{\partial z}{\partial\eta}.$$

By substituting here $\dfrac{\partial z}{\partial\eta}$ we find:

$$\frac{\partial z}{\partial\xi}=-\frac{2u}{Qi(v-u)^2}\cdot\frac{\partial v}{\partial\xi}.$$

We can now write down out of (III) the following set of equations:

$$\left.\begin{aligned}\frac{\partial x}{\partial\xi}&=\frac{1}{2}\left(u-\frac{1}{u}\right)\cdot\frac{\partial z}{\partial\xi}=\frac{-(u^2-1)}{Qi(v-u)^2}\cdot\frac{\partial v}{\partial\xi}\\ \frac{\partial x}{\partial\eta}&=\frac{1}{2}\left(v-\frac{1}{v}\right)\cdot\frac{\partial z}{\partial\eta}=\frac{v^2-1}{Qi(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\\ \frac{\partial y}{\partial\xi}&=\frac{1}{2i}\left(u+\frac{1}{u}\right)\cdot\frac{\partial z}{\partial\xi}=\frac{u^2+1}{Q(v-u)^2}\cdot\frac{\partial v}{\partial\xi}\\ \frac{\partial y}{\partial\eta}&=\frac{1}{2i}\left(v+\frac{1}{v}\right)\cdot\frac{\partial z}{\partial\eta}=\frac{-(v^2+1)}{Q(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\\ \frac{\partial z}{\partial\xi}&= \qquad\qquad\qquad =\frac{-2u}{Qi(v-u)^2}\cdot\frac{\partial v}{\partial\xi}\\ \frac{\partial z}{\partial\eta}&= \qquad\qquad\qquad =\frac{2v}{Qi(v-u)^2}\quad\frac{\partial u}{\partial\eta}\end{aligned}\right\}\quad\ldots\quad(VI)$$

So, as soon as u and v are known, the problem will be solved.

§ 3. In order now to write down the equations which u and v must satisfy, we can make use of (IV) and (VI), or we can use the conditions of integrability.

(IV,a) gives:

$$\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}=\frac{-2u}{iQ(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\,\frac{\partial u}{\partial \eta}+\frac{2v}{iQ}\left(\frac{2}{(v-u)^2}\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{v-u}\frac{\partial^2 u}{\partial \xi \partial \eta}\right).$$

(IV,b) gives:

$$\frac{\partial u}{\partial \eta}\cdot\frac{\partial z}{\partial \xi}=\frac{-2u}{iQ(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\,\frac{\partial u}{\partial \eta}+\frac{2u}{iQ}\left(\frac{2}{(v-u)^2}\cdot\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{v-u}\cdot\frac{\partial^2 u}{\partial \xi \partial \eta}\right).$$

Out of (VI) we find:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial z}{\partial \eta}\right)=-\frac{2}{iQ}\frac{v+u}{(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}+\frac{2v}{iQ(v-u)}\left(\frac{2}{(v-u)^2}\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{v-u}\frac{\partial^2 u}{\partial \xi \partial \eta}\right),$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial z}{\partial \xi}\right)=-\frac{2\,v+u}{iQ(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}+\frac{2u}{iQ(v-u)}\left(\frac{2}{(v-u)^2}\cdot\frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}-\frac{1}{v-u}\frac{\partial^2 v}{\partial \xi \partial \eta}\right),$$

and

$$\frac{\partial^2 z}{\partial \xi \partial \eta}=\frac{Q}{2i}\left(\frac{u}{v}-\frac{v}{u}\right)\cdot\frac{\partial z}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}\text{ gives }\frac{\partial^2 z}{\partial \xi \partial \eta}=-\frac{2\,v+u}{iQ(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}.$$

The equations given above show that all the conditions of the problem can be satisfied in the only way by putting:

$$\frac{2}{(v-u)^2}\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+\frac{1}{(v-u)}\cdot\frac{\partial^2 u}{\partial \xi \partial \eta}=0\text{ and }-\frac{2}{(v-u)^2}\frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}+\frac{1}{(v-u)}\frac{\partial^2 v}{\partial \xi \partial \eta}=0,$$

which equations we write in the form:

$$\left.\begin{aligned}2\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}+(v-u)\frac{\partial^2 u}{\partial \xi \partial \eta}=0\\2\frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}-(v-u)\frac{\partial^2 v}{\partial \xi \partial \eta}=0\end{aligned}\right\}\quad\dots\dots\quad (VII)$$

So the problem is entirely reduced to the integration of these two sumultaneous differential equations which are of order two and non-linear.

It is easy to deduce from (VII), that the conditions

$$\frac{\partial}{\partial \eta}\left(\frac{\partial x}{\partial \xi}\right)=\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \eta}\right)\text{ and }\frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi}\right)=\frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \eta}\right),$$

are satisfied.

We find namely always:

$$\frac{\partial^2 x}{\partial \xi \partial \eta}=-\frac{2(uv-1)}{Qi(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}\quad,\quad\frac{\partial^2 y}{\partial \xi \partial \eta}=\frac{2(uv+1)}{Q(v-u)^3}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial v}{\partial \xi},$$

whilst

$$\frac{\partial^2 z}{\partial\xi\partial\eta} = -\frac{2(v+u)}{iQ(v-u)^3}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi}.$$

After substitution we get:

$$D' = -\frac{H}{2}F \text{ and } X^2 + Y^2 + Z^2 = 1,$$

so that really all the conditions of the problem prove to be satisfied by the equations (VII). Thus only the solution of (VII) is left to be found.

§ 4. We already know, that for the coordinates ξ and η

$$D' = -\frac{H}{2}\cdot F$$

must be satisfied.

But moreover follows from the equations of CODAZZI [1]:

$$\frac{\partial D}{\partial\eta} = 0 \quad\text{and}\quad \frac{\partial D''}{\partial\xi} = 0$$

So

$$D = f_1(\xi) \text{ and } D'' = f_2(\eta), \quad . \quad . \quad . \quad . \quad (VIII)$$

where f_1 and f_2 are respectively functions of ξ and η only.

The case that either D or D'' is equal to zero offers no difficulties, but nothing remarkable either.

The case that D and D'' are both equal to zero, leads, as is immediately clear, to the sphere as the simplest form of a surface with constant mean curvature. We can namely write down the condition for umbilical points, which is as follows with the omission of infinitesimals of higher order: [2])

$$\frac{E}{D} = \frac{F}{D'} = \frac{G}{D''}.$$

When for each point of the surface $E = G = 0$ then each point is an umbilical point, as soon as always $D = D'' = 0$, and these surfaces are (in as far as it concerns the real solution) spheres only.

§ 5. We shall now take the matter a little more generally.

Let us regard the total curvature of a surface as a simultaneous differential-invariant of both groundforms, we then find [3]):

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1) BIANCHI, l. c. p. 91. In using the coordinates ξ and η the CHRISTOFFEL symbols are all zero, except $\left\{\begin{matrix}1 & 1\\ & 1\end{matrix}\right\}$ and $\left\{\begin{matrix}2 & 2\\ & 2\end{matrix}\right\}$. By making use of $D = -\frac{H}{2}F$, we prove what was said in the text.

2) See e g. V. and K. KOMMERELL, Allgemeine Theorie der Raumkurven und Flächen, II, p. 21.

3) BIANCHI, l.c. p. 68.

$$\text{Total curvature} = \frac{DD''-D'^2}{EG-F^2} = \frac{H^2}{4} - \frac{f_1(\xi)f_2(\eta)}{F^2} =$$

$$= \frac{1}{2iF}\frac{\partial}{\partial\eta}\left(\frac{2}{iF}\frac{\partial F}{\partial\xi}\right) = -\frac{1}{F}\frac{\partial^2(lF)}{\partial\xi\partial\eta} \quad (IX)$$

(We notice moreover that, as is directly to be seen,

$$\frac{2}{F} = \frac{1}{r_1} - \frac{1}{r_2},$$

where r_1 and r_2 are the principal radii of curvature).

Let us now deduce from (VI) the value of F, we then find:

$$F = -\frac{2}{Q^2(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi},$$

or:

$$F = \frac{8}{H^2(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi}.$$

We substitute this value of F into (IX) by means of the following calculations. Out of (VII) follows

$$\frac{1}{F}\frac{\partial F}{\partial\eta} = \frac{\dfrac{\partial^2 u}{\partial\eta^2}}{\dfrac{\partial u}{\partial\eta}} + \frac{2}{v-u}\cdot\frac{\partial u}{\partial\eta},$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{F}\frac{\partial F}{\partial\eta}\right) = \frac{\partial}{\partial\xi}\frac{\dfrac{\partial^2 u}{\partial\eta^2}}{\dfrac{\partial u}{\partial\eta}} - 2\,\frac{\dfrac{\partial u}{\partial\xi}\cdot\dfrac{\partial u}{\partial\eta} + \dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}}{(v-u)^2}.$$

This must be equal to

$$-\frac{H^2}{4}F + \frac{f_1(\xi)\cdot f_2(\eta)}{F} = -\frac{2}{(v-u)^2}\cdot\frac{\partial u}{\partial\eta}\cdot\frac{\partial v}{\partial\xi} + \frac{H^2(v-u)^2 f_1(\xi)f_2(\eta)}{8\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}},$$

and so we find:

$$\frac{H^2\cdot f_1(\xi)\cdot f_2(\eta)\cdot(v-u)^2}{8\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}} = \frac{\partial}{\partial\xi}\frac{\dfrac{\partial^2 u}{\partial\eta^2}}{\dfrac{\partial u}{\partial\eta}} + \frac{\dfrac{\partial^2 u}{\partial\xi\,.\,\partial\eta}}{v-u}.$$

The second member can be once more reduced by means of (VII), and we find:

$$\frac{H^2 f_1(\xi)f_2(\eta)(v-u)^2}{8\dfrac{\partial u}{\partial\eta}\cdot\dfrac{\partial v}{\partial\xi}} = \frac{2}{(v-u)^2}\cdot\frac{\partial v}{\partial\eta}\cdot\frac{\partial u}{\partial\xi}.$$

So

$$H^2 f_1(\xi) f_2(\eta) = \frac{16}{(v-u)^4} \frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} \quad . \quad . \quad . \quad (X)$$

§ 6. Let us now return to equation (VII). We see immediately that a solution, which does not cause $F = \frac{8}{H^2 (v-u)^2} \cdot \frac{\partial v}{\partial \xi} \cdot \frac{\partial u}{\partial \eta}$ to vanish, is given by

$$u = \varphi(\eta) \quad , \quad v = \psi(\xi),$$

where φ and ψ are respectively functions of η and ξ only.

It is clear that equation (IX) is satisfied, when $f_1(\xi) = f_2(\eta) = 0$, so when $D = D'' = 0$ (§ 4).

It is worth noticing, that when $u = \varphi(\eta)$ and $v = \psi(\xi)$ are substituted into the equation for F, this form becomes a solution of

$$-\frac{H^2}{4} F = \frac{\partial}{\partial \xi}\left(\frac{1}{F} \cdot \frac{\partial F}{\partial \eta}\right) .$$

and so this tallies perfectly, because we have here the differential equation of LIOUVILLE. Indeed, the problem of the surfaces with constant mean curvature always leads to an extended equation of LIOUVILLE, as (IX) does, in whatever way we treat it.

That we really find a sphere here must follow from (VI). These equations give for $u = \varphi(\eta)$ and $v = \psi(\xi)$,

$$z = \frac{1}{Qi} \frac{v+u}{v-u},$$

$$x = \frac{1}{Qi} \frac{uv-1}{v-u},$$

$$y = -\frac{1}{Q} \frac{uv+1}{v-u},$$

the wellknown formulæ for the sphere in minimal coordinates.

We find·

$$x^2 + y^2 + z^2 = -\frac{1}{Q^2} = \frac{4}{H^2},$$

i.e. a sphere with radius $\frac{2}{H}$, as is necessary.

Now that we have regarded the special case $f_1(\xi) = f_2(\eta) = 0$, we can put both functions equal to 1 by introducing new functions

$$f_1(\xi) = \xi_1 \text{ and } f_2(\eta) = \eta_1,$$

which we shall again indicate by ξ and η. This is of high importance, if eventually the solution of equation (VII) were to be found.

§ 7. We can now put the question whether the equations (VII) can be solved by putting u equal to $f(v)$, where for the present f is arbitrary.

From (VII) can be deduced

$$\frac{\dfrac{\partial^2 v}{\partial\xi\partial\eta}}{\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}+\frac{\dfrac{\partial^2 u}{\partial\xi\partial\eta}}{\dfrac{\partial u}{\partial\xi}\cdot\dfrac{\partial u}{\partial\eta}}=0.$$

For $u=f(v)$ this leads to

$$\frac{\partial u}{\partial\xi}=f'(v)\cdot\frac{\partial v}{\partial\xi},$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=f''(v)\cdot\frac{\partial v}{\partial\xi}\cdot\frac{\partial v}{\partial\eta}+f'(v)\cdot\frac{\partial^2 v}{\partial\xi\partial\eta}.$$

So:

$$\frac{\dfrac{\partial^2 v}{\partial\xi\partial\eta}}{\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}+\frac{f''(v)\cdot\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}+f'(v)\cdot\dfrac{\partial^2 v}{\partial\xi\partial\eta}}{f'^2(v)\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}=0,$$

or

$$\{f'^2(v)+f'(v)\}\frac{\partial^2 v}{\partial\xi\partial\eta}+f''(v)\cdot\frac{\partial v}{\partial\xi}\cdot\frac{\partial v}{\partial\eta}=0.$$

Then, according to (VII), $\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}=\dfrac{v-u}{2}\cdot\dfrac{\partial^2 v}{\partial\xi\partial\eta}.$

So:

$$f'^2(v)+f'(v)+\frac{v-f(v)}{2}f''(v)=0.$$

One integral of this is sufficient to recognize the nature of the surfaces found. We find that satisfies

$$f(v)=-v\,[1].$$

[1]) Prof. W. Kapteyn was so kind as to draw my attention to the following general solution of the differential equation.

Put

$$f(v)=y,$$

then

$$\frac{v-y}{2}\frac{d^2y}{dv^2}+\left(\frac{dy}{dv}\right)^2+\frac{dy}{dv}=0.$$

Now put

$$y=v+w,$$

so

$$\frac{dy}{dv}=1+\frac{dw}{dv}\ ,\quad \frac{d^2y}{dv^2}=\frac{d^2w}{dv^2},$$

so that

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The equations (VII) become

$$\frac{\partial u}{\partial \xi}\cdot\frac{\partial u}{\partial \eta} - u\frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \text{ and } \frac{\partial v}{\partial \xi}\cdot\frac{\partial v}{\partial \eta} - v\frac{\partial^2 v}{\partial\xi\partial\eta} = 0,$$

which are satisfied by a function and its opposite. From this we deduce:

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\,(lu)}{\partial\eta}\right) = 0 .$$

Therefore e. g.

$$u = e^{\psi(\eta)+\varphi(\xi)}\ ,\ v = -\,e^{\psi(\eta)+\varphi(\xi)} .$$

By quadratures we find out of (VI),

$$2Qiz = -\,\psi(\eta) + \varphi(\xi),$$
$$4Qix = \quad e^{\psi(\eta)+\varphi(\xi)} + e^{-\psi(\eta)-\varphi(\xi)},$$
$$4Qy = -\,e^{\psi(\eta)+\varphi(\xi)} + e^{-\psi(\eta)-\varphi(\xi)}.$$

The surface is a cylinder of revolution. Its section with the plane XOY is a circle, as we find

$$y^2 + x^2 = -\frac{1}{4Q^2} = \frac{1}{H^2} .$$

The radius of the circle is therefore $\frac{1}{H}$, as it has to be.

We can furthermore easily show that our solution agrees with the differential equations (IX), when we put

$$f_1(\xi) = f_2(\eta) = 1.$$

We find namely that the second member becomes zero, so that

$$\frac{1}{F} = \frac{H}{2} = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

As moreover $\frac{1}{F} = \frac{1}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$, as we saw before, r_2 is therefore $= \infty$.

§ 8. We can now investigate what in the equations (VII) the significance would be of a solution $u = \chi(\xi)$, if it were possible.

$$-\frac{w}{2}\frac{d^2w}{dv^2} + \left(\frac{dw}{dv}\right)^2 + 3\left(\frac{dw}{dv}\right) + 2 = 0.$$

Let $\frac{dw}{dv} = p$, so $\frac{d^2w}{dv^2} = p\,\frac{dp}{dw}$,

then:

$$-\frac{w}{2}\,p\,\frac{dp}{dw} + (p+1)(p+2) = 0,$$

from which ensues, $\frac{(p+2)^2}{p+1} = kw^2$ ($k =$ const.)

For $k = 0$ this solution gives the one used in the text.

The equation

$$\frac{\partial u}{\partial \xi} \cdot \frac{\partial u}{\partial \eta} + (v-u)\ 2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

is satisfied by $u = \chi(\xi)$.

So there remains to be integrated

$$2\frac{\partial v}{\partial \xi} \cdot \frac{\partial v}{\partial \eta} - (v-u)\frac{\partial^2 v}{\partial \xi \partial \eta} = 0\,,$$

when $u = \chi(\xi)$.

We find:

$$\frac{\partial v}{\partial \xi} = \frac{1}{2}(v - \chi(\xi))^2 \cdot f(\xi),$$

with $f(\xi)$ as arbitrary function of ξ.

The solution $u = \chi(\xi)$ furnishes (see (VI)) the value zero for $\frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}$ and $\frac{\partial z}{\partial \eta}$; whilst for $\frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}$ and $\frac{\partial z}{\partial \xi}$ the wellknown formulæ are found back for the minimal curves.

Entirely the same (with exchange of u and v, ξ and η) is found by putting $v = \chi_1(\eta)$.

This solution therefore shows what relations there are between the minimal surfaces and those under consideration. For the former we have but to join the two solutions found to get the complete solution with two arbitrary functions. So we see that the minimal surfaces are translation surfaces, generated by moving a minimal curve out of a set along the various points of a curve out of the second set; i. o. w. we have found back the integration of the minimal surfaces and in the usual form too.

Because of H tending to zero there is in this case no fear of F becoming 0.

§ 9. Now that the special cases of sphere (plane), cylinder and minimal surfaces are excluded, the integration of the equations (VII) would remain. I have not been able to attain more than the lowering of the order of the two differential equations, which is perhaps a step onward to a complete solution or to solutions for definite series of surfaces.

To this end we put:

$$\frac{\partial v}{d\xi} \cdot \frac{1}{(v-u)^2} = \frac{w_1}{2} \quad , \quad \frac{\partial u}{\partial \eta}\frac{1}{(v-u)^2} = -\frac{w_2}{2},$$

where w_1 and w_2 are functions of ξ and η.

From these we derive, by differentiation with respect to ξ and η respectively

$$\frac{\partial^2 v}{\partial\xi\partial\eta}=(v-u)\cdot w_1\cdot\left(\frac{\partial v}{\partial\eta}-\frac{\partial u}{\partial\eta}\right)+\frac{1}{2}(v-u)^2\cdot\frac{\partial w_1}{\partial\eta},$$

and

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=-(v-u)\cdot w_2\cdot\left(\frac{\partial v}{\partial\xi}-\frac{\partial u}{\partial\xi}\right)-\frac{1}{2}(v-u)^2\cdot\frac{\partial w_2}{\partial\xi}.$$

By means of the two non-differentiated equations and by equation (VII), we deduce from our last equation:

$$w_1\cdot\frac{\partial v}{\partial\eta}=w_1\left(\frac{\partial v}{\partial\eta}-\frac{\partial u}{d\eta}\right)+\frac{1}{2}(v-u)\cdot\frac{\partial w_1}{\partial\eta}$$

$$\text{and}\quad -w_2\cdot\frac{\partial u}{\partial\xi}=w_2\left(\frac{\partial v}{\partial\xi}-\frac{\partial u}{\partial\xi}\right)+\frac{1}{2}(v-u)\cdot\frac{\partial w_2}{\partial\xi},$$

or:

$$w_1\frac{\partial u}{\partial\eta}=\frac{1}{2}(v-u)\cdot\frac{dw_1}{\partial\eta}\quad\text{and}\quad w_2\frac{\partial v}{\partial\xi}=-\frac{1}{2}(v-u)\cdot\frac{\partial w_2}{\partial\xi},$$

from which ensues:

$$-w_1w_2(v-u)=\frac{\partial w_1}{\partial\eta}\quad\text{and}\quad -w_1w_2(v-u)=\frac{\partial w_2}{\partial\xi}.$$

So we may put:

$$w_1=\frac{\partial f}{\partial\xi}\quad\text{and}\quad w_2=\frac{\partial f}{\partial\eta},$$

where f is a function of ξ and η which has however to satisfy a new differential equation.

So we have:

$$\frac{\partial v}{\partial\xi}\cdot\frac{1}{(v-u)^2}=\frac{1}{2}\frac{\partial f}{\partial\xi}\quad\text{and}\quad\frac{\partial u}{\partial\eta}\cdot\frac{1}{(v-u)^2}=-\frac{1}{2}\frac{\partial f}{\partial\eta},$$

whilst moreover:

$$v-u=-\frac{\dfrac{\partial^2 f}{\partial\xi\partial\eta}}{\dfrac{\partial f}{\partial\xi}\cdot\dfrac{\partial f}{\partial\eta}}.$$

Out of (VII) follows:

$$v-u=2\frac{\dfrac{\partial v}{\partial\xi}\cdot\dfrac{\partial v}{\partial\eta}}{\dfrac{\partial^2 v}{\partial\xi\partial\eta}}\quad\text{and}\quad v-u=-2\frac{\dfrac{\partial u}{\partial\xi}\cdot\dfrac{\partial u}{\partial\eta}}{\dfrac{\partial^2 u}{\partial\xi\partial\eta}}.$$

By substitution of $v-u$, $\frac{\partial v}{\partial \xi}$ and $\frac{\partial v}{\partial \eta}$ we thus find:

$$1=\frac{(v-u)\frac{\partial f}{\partial \xi}\cdot\frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial \xi \partial \eta}}, \qquad 1=\frac{(v-u)\frac{\partial f}{\partial \eta}\cdot\frac{\partial u}{\partial \xi}}{\frac{\partial^2 u}{\partial \xi \partial \eta}}$$

$$1=-\frac{\frac{\partial^2 f}{\partial \xi \partial \eta}\cdot\frac{\partial v}{\partial \eta}}{\frac{\partial^2 v}{\partial \xi \partial \eta}\cdot\frac{\partial f}{\partial \eta}}, \qquad 1=-\frac{\frac{\partial^2 f}{\partial \xi \partial \eta}\cdot\frac{\partial u}{\partial \xi}}{\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial f}{d \xi}}.$$

$$\frac{\frac{\partial^2 v}{\partial \xi \partial \eta}}{\frac{\partial v}{\partial \eta}}=-\frac{\frac{\partial^2 f}{\partial \xi \partial \eta}}{\frac{\partial f}{\partial \eta}}, \qquad \frac{\frac{\partial^2 u}{\partial \xi \partial \eta}}{\frac{\partial u}{\partial \xi}}=-\frac{\frac{\partial^2 f}{\partial \xi \partial \eta}}{\frac{\partial f}{\partial \xi}}.$$

After integration we find:

$$\frac{\partial v}{\partial \eta}\cdot\frac{\partial f}{\partial \eta}=F_2(\eta) \text{ and } \frac{\partial u}{\partial \xi}\cdot\frac{\partial f}{\partial \xi}=F_1(\xi).$$

Joining these equations to the values of $\frac{\partial v}{\partial \xi}$ and $\frac{\partial u}{\partial \eta}$, we find:

$$\frac{\partial v}{\partial \eta}\cdot\frac{\partial u}{\partial \eta}=-\frac{1}{2}(v-u)^2 F_2(\eta) \text{ and } \frac{\partial u}{\partial \xi}\cdot\frac{\partial v}{\partial \xi}=\frac{1}{2}(v-u)^2 F_1(\xi).$$

These equations must be regarded as the intermediate integrals; they contain the arbitrary functions F_2 and F_1, and it is easy to prove that by differentiation they lead back to the two equations (VII) of order two.

It goes almost without saying that F_2 and F_1 appearing here are closely connected to f_1 and f_2 appearing in (VIII).

From the equations just found follows:

$$\frac{\partial v}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial u}{\partial \xi}=-\frac{1}{4}(v-u)^4 . F_2(\eta) . F_1(\xi),$$

or

$$F_2(\eta) . F_1(\xi)=-\frac{4\frac{\partial v}{\partial \eta}\cdot\frac{\partial v}{\partial \xi}\cdot\frac{\partial u}{\partial \eta}\cdot\frac{\partial u}{\partial \xi}}{(v-u)^4}.$$

If we compare this to (X), then:

$$-4F_2(\eta) . F_1(\xi)=H^2 f_1(\xi) f_2(\eta).$$

The first integrals found satisfy therefore all the conditions entirely. We have transformed our original coordinates in such a way that

$f_1(\xi)$ and $f_2(\eta)$ both became 1 and so now we can take in accordance with it:

$$F_2(\eta) = \frac{H}{2i} \text{ and } F_1(\xi) = \frac{H}{2i},$$

so that the first integrals become:

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} = -\frac{H}{4i}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = \frac{H}{4i}(v-u)^2,$$

or

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \eta} = \frac{Q}{2}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = -\frac{Q}{2}(v-u)^2 \quad . \quad . \quad . \quad (A)$$

By replacing moreover $v-u$ by s_1 and $v+u$ by s_2 the final equations become:

$$\left(\frac{\partial s_1}{\partial \eta}\right)^2 = \left(\frac{\partial s_2}{\partial \eta}\right)^2 - 2\,Q\,s_1^2 \text{ and } \left(\frac{\partial s_1}{\partial \xi}\right)^2 = \left(\frac{d s_2}{\partial \xi}\right)^2 + 2\,Q\,s_1^2 \quad . \quad (B)$$

These are still to be solved.

Mathematics. — *"On the multiplication of trigonometrical series."* By Prof. W. KAPTEYN,

1. If $f(x)$ and $\varphi(x)$ are two functions which are finite and continuous in the interval from $x=0$ to $x=\pi$, we have

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots.$$
$$f(x) = b_1 \sin x + b_2 \sin 2x + \ldots.$$
$$\varphi(x) = \tfrac{1}{2}a'_0 + a'_1 \cos x + a'_2 \cos 2x \ldots.$$
$$\varphi(x) = b'_1 \sin x + b'_2 \sin 2x + \ldots.$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(\omega) \cos n\omega \, d\omega \qquad b_n = \frac{2}{\pi}\int_0^\pi f(\omega) \sin n\omega \, d\omega$$

$$a'_n = \frac{2}{\pi}\int_0^\pi \varphi(\omega) \cos n\omega \, d\omega \qquad b'_n = \frac{2}{\pi}\int_0^\pi \varphi(\omega) \sin n\omega \, d\omega.$$

In the same way the product $f(x) \,.\, \varphi(x)$ may be developed, this product being finite and continuous in the same interval; therefore

$$f(x) \,.\, \varphi(x) = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + \ldots.$$
$$f(x) \,.\, \varphi(x) = B_1 \sin x + B_2 \sin 2x + \ldots.$$

where

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega)\,\varphi(\omega) \cos n\omega \, d\omega, \qquad B_n = \frac{2}{\pi}\int_0^\pi f(\omega)\,\varphi(\omega) \sin n\omega \, d\omega.$$