$f_1(\xi)$ and $f_2(\eta)$ both became 1 and so now we can take in accordance with it:

$$F_2(\eta) = \frac{H}{2i} \text{ and } F_1(\xi) = \frac{H}{2i},$$

so that the first integrals become:

$$\frac{\partial v}{\partial \eta} \cdot \frac{\partial u}{\partial \eta} = -\frac{H}{4i}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = \frac{H}{4i}(v-u)^2,$$

or

$$\frac{\partial u}{\partial \eta} \cdot \frac{\partial v}{\partial \eta} = \frac{Q}{2}(v-u)^2 \text{ and } \frac{\partial u}{\partial \xi} \cdot \frac{\partial v}{\partial \xi} = -\frac{Q}{2}(v-u)^2 \quad . \quad . \quad . \quad (A)$$

By replacing moreover $v - u$ by s_1 and $v + u$ by s_2 the final equations become:

$$\left(\frac{\partial s_1}{\partial \eta}\right)^2 = \left(\frac{\partial s_2}{\partial \eta}\right)^2 - 2\,Q\,s_1^2 \text{ and } \left(\frac{\partial s_1}{\partial \xi}\right)^2 = \left(\frac{d s_2}{\partial \xi}\right)^2 + 2\,Q\,s_1^2 \quad . \quad (B)$$

These are still to be solved.

Mathematics. — *"On the multiplication of trigonometrical series."* By Prof. W. KAPTEYN,

1. If $f(x)$ and $\varphi(x)$ are two functions which are finite and continuous in the interval from $x = 0$ to $x = \pi$, we have

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots.$$
$$f(x) = b_1 \sin x + b_2 \sin 2x + \ldots.$$
$$\varphi(x) = \tfrac{1}{2}a'_0 + a'_1 \cos x + a'_2 \cos 2x \ldots.$$
$$\varphi(x) = b'_1 \sin x + b'_2 \sin 2x + \ldots.$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(\omega) \cos n\omega \, d\omega \qquad b_n = \frac{2}{\pi}\int_0^\pi f(\omega) \sin n\omega \, d\omega$$

$$a'_n = \frac{2}{\pi}\int_0^\pi \varphi(\omega) \cos n\omega \, d\omega \qquad b'_n = \frac{2}{\pi}\int_0^\pi \varphi(\omega) \sin n\omega \, d\omega.$$

In the same way the product $f(x) \cdot \varphi(x)$ may be developed, this product being finite and continuous in the same interval; therefore

$$f(x) \cdot \varphi(x) = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + \ldots.$$
$$f(x) \cdot \varphi(x) = B_1 \sin x + B_2 \sin 2x + \ldots.$$

where

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega)\,\varphi(\omega) \cos n\omega \, d\omega, \qquad B_n = \frac{2}{\pi}\int_0^\pi f(\omega)\,\varphi(\omega) \sin n\omega \, d\omega.$$

We shall now investigate the relations which exist between the integrals A_n, B_n and the coefficients a_n, b_n, a'_n, b'_n.

Substituting in A_n for $\varphi(\omega)$ the series of cosines, we obtain

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega) \cos n\omega \, [\tfrac{1}{2} a'_0 + a'_1 \cos \omega + a'_2 \cos 2\omega + \ldots] \, d\omega$$

$$= \tfrac{1}{2} a'_0 \, a_n + \sum_1^\infty \frac{a'_m}{2} \cdot \frac{2}{\pi}\int_0^\pi f(\omega) \, [\cos (m+n)\,\omega + \cos (m-n)\,\omega] \, d\omega$$

$$= \tfrac{1}{2} a'_0 \, a_n + \sum_1^\infty \frac{a'_m}{2} (a_{m+n} + a_{m-n}).$$

This equation may be written in another form; for, because $a'_{-p} = a'_p$,

$$\sum_1^\infty \frac{a'_m}{2} a_{m-n} = \sum_1^\infty \frac{a'_m}{2} a_{n-m} + \tfrac{1}{2} \sum_{n+1}^\infty a'_m \, a_{m-n}$$

or, putting $m + n$ instead of m in the summation from $n + 1$ to ∞

$$\sum_1^\infty \frac{a'_m}{2} a_{m-n} = \tfrac{1}{2} \sum_1^n a'_m \, a_{n-m} + \tfrac{1}{2} \sum_1^\infty a_m \, a'_{m+n} .$$

Hence

$$A_n = \tfrac{1}{2} \sum_0^n a'_m \, a_{n-m} + \tfrac{1}{2} \sum_1^\infty (a'_m \, a_{m+n} + a_m \, a'_{m+n}) \quad . \quad . \quad . \quad (I)$$

If now we substitute in A_n for $\varphi(\omega)$ the series of sines, we have

$$A_n = \frac{2}{\pi}\int_0^\pi f(\omega) \cos n\,\omega \, [b'_1 \sin \omega + b'_2 \sin 2\omega + \ldots] \, d\omega$$

$$= \sum_1^\infty \frac{b'_m}{2} \cdot \frac{2}{\pi}\int_0 f(\omega) \, [\sin (m+n)\,\omega + \sin (m-n)\,\omega] \, d\omega$$

$$= \sum_1^\infty \frac{b'_m}{2} (b_{m+n} + b_{m-n})$$

or, as $b'_{-p} = - bp$

$$A_n = - \tfrac{1}{2} \sum_1^n b'_m \, b_{n-m} + \tfrac{1}{2} \sum_1^\infty (b'_m \, b_{m+n} + b_m \, b'_{m+n}) . \quad . \quad . \quad (II)$$

In the same way we find

$$B_n = \frac{2}{\pi}\int_0^\pi f(\omega) \sin n\omega \, [\tfrac{1}{2} a'_0 + a'_1 \cos \omega + a'_2 \cos 2\omega + \ldots] \, d\omega$$

$$= \tfrac{1}{2} a'_0 \, b_n + \sum^\infty \frac{a'_m}{2} (b_{m+n} - b_{m-n})$$

or, after a slight reduction

$$B_n = \tfrac{1}{2}\sum_0^n a'_m b_{n-m} + \tfrac{1}{2}\sum_1^\infty (a'_m b_{m+n} - b_m a'_{m+n}) \quad . \quad . \quad . \quad . \quad . \quad (III)$$

and

$$B_n = \frac{2}{\pi}\int_0^\pi f(\omega)\, sin\, n\omega\, [b'_1 \, sin\, \omega + b'_2\, sin\, 2\omega + ..]\, d\omega$$

$$= \sum_1^\infty \frac{b'_m}{2} (a_{m-n} - a_{m+n})$$

or

$$B_n = \tfrac{1}{2}\sum_1^n b'_m a_{n-m} + \tfrac{1}{2}\sum_1^\infty (a_m b'_{m+n} - b'_m a_{m+n}) \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

2. If we suppose

$$f^2(x) = \tfrac{1}{2}\mathfrak{A}_0 + \mathfrak{A}_1 \cos x + \mathfrak{A}_2 \cos 2x + \ldots$$
$$= \mathfrak{C}_1 \sin x + \mathfrak{C}_2 \sin 2x + \ldots$$

the four preceding equations give immediately, by putting $\varphi(x) = f(x)$

$$\mathfrak{A}_n = \tfrac{1}{2}\sum_0^n a_m a_{n-m} + \sum_1^\infty a_m a_{m+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\mathfrak{A}_n = -\tfrac{1}{2}\sum_1^n b_m b_{n-m} + \sum_1^\infty b_m b_{m+n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\mathfrak{C}_n = \tfrac{1}{2}\sum_0^n a_m b_{n-m} + \tfrac{1}{2}\sum_1^\infty (a_m b_{m+n} - b_m a_{m+n}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

3. From the four equations of Art. 1, the beautiful theorem of Parseval may be easily deduced. For, supposing that for all the values on the circumference of the circle $mod\, z = 1$, we have

$$\tfrac{1}{2}a_0 + a_1 z + a_2 z^2 + \ldots = \varphi(z)$$
$$\tfrac{1}{2}a'_0 + \frac{a'_1}{z} + \frac{a'_2}{z^2} + \ \ldots = \psi(z),$$

it is evident, if we assume in succession $z = e^{i\omega}$ and $z = e^{-i\omega}$, that

$$F_1(\omega) + i F_2(\omega) = \varphi(e^{i\omega}) \quad G_1(\omega) - i G_2(\omega) = \psi(e^{i\omega})$$
$$F_1(\omega) - i F_2(\omega) = \varphi(e^{-i\omega}) \quad G_2(\omega) + i G_2(\omega) = \psi(e^{-i\omega}).$$

Multiplying these equations and adding the results we obtain

$$2\,[F_1(\omega)\, G_1(\omega) + F_2(\omega)\, G_2(\omega)] = \varphi(e^{i\omega})\,\psi(e^{i\omega}) + \varphi(e^{-i\omega})\,\psi(e^{-i\omega})$$

where

$$F_1(\omega) = F_1 = \tfrac{1}{2}a_0 + a_1 \cos\omega + a_2 \cos 2\omega + \ldots$$
$$G_1(\omega) = G_1 = \tfrac{1}{2}a'_0 + a'_1 \cos\omega + a'_2 \cos 2\omega + \ldots$$
$$F_2(\omega) = F_2 = a_1 \sin\omega + a_2 \sin 2\omega + \ldots$$
$$G_2(\omega) = G_2 = a'_1 \sin\omega + a'_2 \sin 2\omega + \ldots$$

If now we put $n=0$ in the equations (I) and (II) we find

$$\frac{2}{\pi}\int_0^\pi F_1\, G_1\, d\omega = \tfrac{1}{2}\, a_0\, a'_0 + a_1\, a'_1 + a_2\, a'_2 + \ldots .$$

$$\frac{2}{\pi}\int_0^\pi F_2\, G_2\, d\omega = \qquad\qquad a_1\, a'_1 + a_2\, a'_2 + \ldots$$

thus

$$\frac{1}{2\pi}\int_0^\pi \{\varphi\,(e^{i\omega})\,\psi\,(e^{i\omega}) + \varphi\,(e^{-i\omega})\,\psi\,(e^{-i\omega})\}\, d\omega = \frac{1}{4}\, a_0 a'_0 + a_1 a'_1 + a_2 a'_2 + \ldots$$

which is the theorem in question.

4. From the preceding formulae we may also deduce the values of several interesting series. For, if the series for $f(x)$ and $\varphi(x)$ are given, and the integrals

$$\int_0^\pi f(\omega)\,\varphi\,(\omega)\cos n\omega\, d\omega \text{ and} \int_0^\pi f(\omega)\,\varphi\,(\omega)\sin n\omega\, d\omega$$

are to be found, the values of the series in the second members of the given equations may be determined. To show this, we shall make the following application of the formulae (1), (2) and (3).

Suppose $f(x) = x$, then

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} - \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \ldots .\right)$$

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots .\right)$$

and

$$\mathfrak{A}_n = \frac{2}{\pi}\int_0^\pi \omega^2 \cos n\omega\, d\omega = 4\,\frac{\cos n\pi}{n^2}$$

$$\mathfrak{C}_n = \frac{2}{\pi}\int_0^\pi \omega^2 \sin n\omega\, d\omega = -\frac{2\pi\cos n\pi}{n} - \frac{4\,(1-\cos n\pi)}{\pi n^3}.$$

Now the formula (1) gives, because

$$a_2 = a_4 = a_6 = \ldots = 0$$

$$\mathfrak{A}_0 = \tfrac{1}{2}\, a_0^2 + a_1^2 + a_3^2 + a_5^2 + \ldots$$

$$\mathfrak{A}_2 = \tfrac{1}{2}\, a_1^2 + a_1 a_3 + a_3 a_5 + a_5 a_7 + \ldots$$

$$\mathfrak{A}_4 = a_1 a_3 + a_1 a_5 + a_3 a_7 + a_5 a_9 + \ldots$$

$$\mathfrak{A}_6 = a_1 a_5 + \frac{a_3^2}{2} + a_1 a_7 + a_3 a_9 + a_5 a_{11} + \ldots$$

. . . .

therefore

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots \qquad =\frac{\pi^4}{96}$$

$$\frac{1}{1^2.3^2}+\frac{1}{3^3.7^2}+\frac{1}{5^2.9^2}+\dots=\frac{\pi^2}{16}-\frac{1}{2}$$

$$\frac{1}{1^2.5^2}+\frac{1}{3^2.7^2}+\frac{1}{5^2.9^2}+\dots=\frac{\pi^2}{64}-\frac{1}{9}$$

$$\frac{1}{1^2.7^2}+\frac{1}{3^2.9^2}+\frac{1}{5^2.11^2}+\cdot=\frac{\pi^2}{144}-\frac{137}{4050}$$

. . . .

According to formula (2) we have

$$\mathfrak{A}_0=\sum_1^\infty b_m^2$$

$$\mathfrak{A}_1=\sum_1^\infty b_m b_{m+1}$$

$$\mathfrak{A}_2=-\tfrac{1}{2}\,b_1^2+\sum_1^\infty b_m b_{m+2}$$

$$\mathfrak{A}_3=-\,b_1 b_2+\sum_1^\infty b_m b_{m+3}$$

$$\mathfrak{A}_4=-\tfrac{1}{2}\,(2b_1b_3+b_2^2)+\sum_1^\infty b_m b_{m+4}$$

$$\mathfrak{A}_5=-\,(b_1b_4+b_2b_3)+\sum_1^\infty b_m b_{m+5}$$

$$\mathfrak{A}_6=-\tfrac{1}{2}\,(2b_1b_5+2b_2b_4+b_3^2)+\sum_1^\infty b_m b_{m+6}$$

. . . .

From which may be deduced

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6}$$

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots=1$$

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\dots=\frac{3}{4}$$

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\dots=\frac{11}{18}$$

$$\frac{1}{1.5}+\frac{1}{2.6}+\frac{1}{3.7}+\dots=\frac{25}{48}$$

$$\frac{1}{1.6}+\frac{1}{2.7}+\frac{1}{3.8}+\dots=\frac{137}{300}$$

$$\frac{1}{1.7}+\frac{1}{2.8}+\frac{1}{3.9}+\dots=\frac{49}{120}$$

. . . .

In the same way the formula (3) gives

$$\mathfrak{C}_1 = \tfrac{1}{2}\, a_0 b_1 + \tfrac{1}{2}\,(a_1 b_2 - b_2 a_3 + a_3 b_4 - b_4\, a_6 + \ldots)$$

$$\mathfrak{C}_2 = \tfrac{1}{2}\,(a_0 b_2 + a_1 b_1) + \tfrac{1}{2}\,(a_1 b_3 - b_1 a_3 + a_3 b_5 - b_3 a_5 + a_5 b_7 - b_5 a_7 + \ldots)$$

$$\mathfrak{C}_3 = \tfrac{1}{2}\,(a_0 b_3 + a_1 b_2) + \tfrac{1}{2}\,(a_1 b_4 - b_2 a_5 + a_3 b_6 - b_4 a_7 + a_5 b_8 - b_6 a_9 + \ldots)$$

$$\mathfrak{C}_4 = \tfrac{1}{2}\,(a_0 b_4 + a_1 b_3 + a_3 b_1) + \tfrac{1}{2}\,(a_1 b_5 - b_1 a_5 + a_3 b_7 - b_3 a_7 + a_5 b_9 - b_5 a_9 + \ldots)$$

$$\mathfrak{C}_5 = \tfrac{1}{2}\,(a_0 b_5 + a_1 b_4 + a_3 b_2) + \tfrac{1}{2}\,(a_1 b_6 - b_2 a_7 + a_3 b_8 - b_4 a_9 + a_5 b_{10} - b_6 a_{11} + \ldots)$$

$$\mathfrak{C}_6 = \tfrac{1}{2}\,(a_0 b_6 + a_1 b_5 + a_3 b_3 + a_5 b_1) + \tfrac{1}{2}\,(a_1 b_7 - b_1 a_7 + a_3 b_9 - b_3 a_9 + a_5 b_{11} - b_5 a_{11} + \ldots)$$

. . . .

from which the following relations may be obtained

$$\frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} + \ldots = \frac{\pi^2}{16} - \frac{1}{2}$$

$$\frac{1}{4.1^2} - \frac{1}{2.5^2} + \frac{1}{6.3^2} - \frac{1}{4.7^2} + \ldots = \frac{\pi^2}{12} - \frac{31}{54}$$

$$\frac{1}{5.1^2} - \frac{1}{1.5^2} + \frac{1}{7.3^2} - \frac{1}{3.7^2} + \ldots = \frac{\pi^2}{16} - \frac{4}{9}$$

$$\frac{1}{6.1^2} - \frac{1}{2.7^2} + \frac{1}{8.3^2} - \frac{1}{4.9^2} + \ldots = \frac{7\pi^2}{60} - \frac{347}{900}$$

$$\frac{1}{7.1^2} - \frac{1}{1.7^2} + \frac{1}{9.3^2} - \frac{1}{3.9^2} + \ldots = \frac{\pi^2}{24} - \frac{187}{675}$$

. . . .

Chemistry. — *"On a crystallised d. fructose tetracetate"*, by Dr. D. H. BRAUNS. (Communicated by Prof. A. P. N. FRANCHIMONT).

Very few crystallised derivatives of *d.* fructose have as yet been obtained. A pentacetate was described by ERWIGS and KOENIGS as a gummy substance. A number of researches have shown, however, that the high temperature at which the reactions generally took place causes a conversion or decomposition of the fructose. As no satisfactory results were obtained with acetic anhydride and acetyl chloride acetyl bromide was employed which reacts at a comparatively low temperature. The greatest possible precautions were taken to exclude moisture and to let the reaction take place at a low temperature. The details will be published in full later on.

Refrigerated d. fructose in fine powder was mixed with a little more than 5 mols. of acetyl bromide at – 15° and after starting the reaction by touching one spot with a tube having the ordinary