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Astronomy. — *"On the mean star-density at different distances from the solar system."* By J. C. KAPTEYN.

(Communicated in the meeting of February 29, 1908).

In the meeting of April 20, 1901, I derived not only the so-called *luminosity-curve* but also the law according to which the star-density, i. e. the number of stars per unit of volume, diminishes with increasing distance from the solar system[1]). I assumed, and the assumption will again be made in the present paper, that there is no absorption of light in space. I then pointed out that the luminosity-curve is not very appreciably modified if we change, within admissable limits the data from which it was derived. On the other hand it was expressly stated that the determination of the change of density was only quite *provisional.* Its discussion was deferred to a subsequent communication, in which, along with the data then used, other elements might be taken into account. (l.c. p. 731).

These other elements are mainly: the total numbers of stars of different magnitude and their mean parallaxes. As to the first, the numbers: a short time ago I treated all the materials accessible to me (see Publications of the Astr. Lab. at Groningen N°. 18) and I think that I obtained very reliable results for the stars brighter than 11.5, fairly reliable ones down to the 15th magnitude. Now, though the mean parallaxes are still wanting, we are already able, by the numbers alone, to arrive at a considerable improvement in the distribution of the densities, at least for the larger distances. Such a derivation will be given in what follows.

As formerly a separate treatment of the regions of different galactic latitude was not yet attempted, because I think that it will be desirable that in carrying out such a separate treatment we investigate at the same time whether it be admissable or not to assume the same luminosity curve (mixture-law) for the different parts of the system.

Already I have collected fairly extensive materials for this purpose, but still some time will have to elapse before the investigation can be carried out with advantage.

In the main my purpose, in making the following determination, was simply to get first notions about the star-density at still greater distances than could be reached in our former investigation. The determination embraces also the smaller distances but it remains to be seen whether for these the correction found is or is not an improvement.

In the *Astron. Journ.* N°. 566 I derived analytical expressions,

1) See also: Publications of the Astr. Lab. at Groningen No. 11.

which fairly represent the numbers found in the communication just mentioned.

Let

$\delta = 2\,512 \ldots = \dfrac{\textit{Apparent brightness of a star of mag } m}{\textit{,, \quad ,, \quad ,, ,, ,, ,, ,, } m+1}$ $(log.\ \delta = 0.4)$;

ϱ = distance from the solar system ($\varrho = 1$ for parallax $= 0''1$);

N_m = number of stars in the whole of the sky between the apparent magnitudes $m - \frac{1}{2}$ and $m + \frac{1}{2}$.

$\Delta(\varrho)$ = star-density = number of stars per unit of volume at distance ϱ (unit of volume = cube, each side of which = unit of distance).

h_m = apparent brightness of a star of the magnitude m ($h_{5.5} = 1$).

L = Luminosity = total quantity of light emitted ($L = 1$ for Sun).

$\varphi(L)\,dL$ = probability that the luminosity of a star, chosen at random, is contained between L and $L + dL$.

$\psi(L) = \int_{\frac{L}{\sqrt{\delta}}}^{L\sqrt{\delta}} \varphi(z)\,dz$ = probability that the luminosity is contained between the limits $L \pm \frac{1}{2}$ mag.

Now, if we assume that $\varphi(L)$ is not dependent on ϱ, we shall have

$$N_m = 4\pi \int_0^\infty \varrho^4\,\Delta(\varrho)\,d\varrho \int_{\frac{h_m}{\sqrt{\delta}}}^{h_m\sqrt{\delta}} \varphi(h\varrho^2)\,d\varrho = 4\pi \int_0^\infty \varrho^2\,\Delta(\varrho)\;\psi(h_m\varrho^2)\,d\varrho \quad (1)$$

The expressions derived in *Astron. Journ.* N°. 566 are:

$$\psi(L) = \frac{\alpha^2\,.\,mod.}{\sqrt{\pi}\,L}\,e^{-\alpha^2[log\,L - T]^2} \quad \ldots\ldots \quad (2)$$

$$\frac{\Delta(\varrho)}{\Delta(0)} = e^{-\beta\varrho} + \beta\varrho\,e^{-\gamma\varrho^2} \quad \ldots \quad (3)$$

in which

$$\left.\begin{aligned} T &= 1.400 \\ \alpha^2 &= 0.385 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

$$\Delta(0) = 111.0 \quad \ldots\ldots\ldots \quad (5)$$

$$\left.\begin{aligned} \beta &= 0.0220 \\ \gamma &= 0.0052 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

In a subsequent part of the same paper, the *numbers* of the stars as given by Pickering led to a new value of $\Delta(0)$ *viz.*

$$\Delta(0) = 136.9 \quad \ldots\ldots\ldots \quad (7)$$

the difference of this value and the value (5) is wholly explained by the constant difference of the photometric scale of Potsdam, which wa

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used for the determination (5) and that of Harvard which served for the derivation of (7).

In what follows the magnitudes have also be reduced to the Harvard scale. I have adopted the luminosity-curve (2), in which the constants have the values (4), without any change. For the density curve (3), however a new determination was obtained by the aid of the total numbers of the stars of different apparent magnitude. In other words: by the aid of formula (1) I derived Δ as a function of ϱ from the *given* values N_m ($m = 2$ to 15) and the *given* form of ψ.

The introduction of the analytical functions (2) and (3) has the advantage of greatly facilitating the computations. Of course we have not to forget, however, that they can be relied on only just to the same extent as that for which we possess observational data. For the luminosity-curve, with the exception only of the stars belonging to the classes of the very greatest apparent brightness, the unlimited use of the formula will not easily give rise to appreciable errors, because extrapolation is only necessary for a very small fraction of the total. On the contrary, the density-curve (3) (which, as we already remarked, is not very accurately determined) furnishes values, which, for ϱ exceeding 60, are to be considered as wholly obtained by *extrapolation.* It will appear from what follows that up to $\varrho = 60$ the values derived from the new materials do not differ from those formerly obtained more than seems in accordance with their uncertainty. That on the other hand, the values for $\varrho > 60$, which we may extrapolate by means of formula (3), are *far* too small; to such an extent that for these greater distances the formula is evidently quite unsatisfactory.

To begin with, I ascertained how the formula (3), in which the constants have the values (6) and (7), represents the N_m of publication 18. A table of the integrals entering in the formula (1) has been given in *Astronomical Journal* N°. 566 for values of m between 0 and 11. (table III) [1]).

[1]) In the calculation of the values of T_1 and T_3 a mistake has been discovered:

			T_1					T_3		
For $m =$	3 0,	instead of	9.12	read	9.13					
	4.0	„ „	12.69	„	12.71					
	8.0	„ „	17.04	„	17.10					
	6.0	„ „	21.86	„	21.99					
	7.0	„ „	26.63	„	26.88					
	8.0	„ „	30.54	„	30.96					
	9.0	„ „	32.80	„	33.42	Instead	of	1.71	read	1.72
	10.0	„ „	32.84	„	33.64	„	„	1.37	„	1.38
	11.0	„ „	30.54	„	31.51	„	„	1.04	„	1.05

For $m = 14$ the values were now expressly computed. The result is as follows:

Total number of stars.

m	Obs. (Publ. 18)	Comp. I. (by Form. (3))	Comp. II.	
4.5 tot 5.5	1 848	1 897	1 788	
6.5 „ 7.5	17 940	18 420	18 650	
8.5 „ 9.5	159 200	140 200	169 500	. . (8)
10.5 „ 11.5	1 275 000	808 200	1 335 000	
13.5 „ 14.5	23 680 000	6 500 000	20 800 000	

The deviation increases strongly with diminishing brightness and is excessive for magnitude 14. We conclude at once, that for the greater distances the formula (3) furnishes a value of the star-density which is much too small. Calculation shows that some approximate agreement is already obtained if we take the stars between $\varrho = 140$ and $\varrho = \infty$ to be 21 times more numerous.

As it thus appears that formula (3) is useless for considerable values of ϱ, I began by retaining that formula exclusively for the values of ϱ below 70 whereas for the values exceeding 70 I assumed that the density diminishes regularly (linearly) from 0.214 to *zero*.

It was easily ascertained that, if we choose the decrease of the density in such a way that it vanishes for $\varrho = 557$, we get considerably nearer to the truth, especially if we take:

$$\Delta_0 = 125.$$

The values obtained in this way were put down in the above table under the head Comp. II.

As a further approximation I also derived corrections for the star-density at distances below 70. It appeared that the results become more satisfactory if the linear decrease of the densities is assumed to begin for distances somewhat smaller than 70.

Having obtained this result I have no further continued these approximations, but I have given up the formula (3) altogether and have tried to determine the luminosity-curve directly in the assumption that, for the intervals between $\varrho = 0$ and $\varrho = 10$; $\varrho = 10$ and $\varrho = 30$; $\varrho = 30$ and 50; $\varrho = 50$ and $\varrho = g$ the density changes linearly in such a way that it vanishes for $\varrho = g$.

In this way the problem is reduced to the derivation of the 5 unknown quantities:

$$\Delta(0) \ ; \ \Delta(10) \ ; \ \Delta(30) \ ; \ \Delta(50) \ ; \ g.$$

For reasons given in the paper quoted above we have to assume that $\frac{\partial \Delta}{\partial \varrho}$ vanishes for $\varrho = 0$. As a consequence $\Delta(10)$ will certainly be little different from $\Delta(0)$. Therefore, in order to reduce the number of unknown quantities as much as possible, I took, in agreement with what was formerly found:

$$\Delta(10) = 0.97\, \Delta(0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The number of unknown quantities is thus lowered to 4.

Putting

$$\frac{\Delta(\varrho)}{\Delta(0)} = D_\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

we have

$$D_\rho = A\varrho + B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in which, for the several intervals, A and B have the following values:

	A	B
$\varrho =$ 0 *to* 10	$\frac{1}{10} D_{10} - \frac{1}{10}$	$1. -$
10 ,, 30	$\frac{1}{20} D_{30} - \frac{1}{20} D_{10}$	$\frac{3}{2} D_{10} - \frac{1}{2} D_{30}$
30 ,, 50	$\frac{1}{20} D_{50} - \frac{1}{20} D_{30}$	$\frac{5}{2} D_{30} - \frac{3}{2} D_{50}$
50 ,, g	$-\frac{1}{g-50} D_{50}$	$\frac{g}{g-50} D_{50}$

. . (12)

The practical advantage of the present form is that the expression (1) for N_m can now be reduced to the well known integral

$$\Theta(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-x^2} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Numerical integration is thus avoided and we obtain relations which are linear in respect to the unknown quantities D_{30}, D_{50} and $\frac{1}{\Delta(0)}$.

If we denote by $(N_m)_0^\rho$ the number of stars between apparent magnitude $m - \frac{1}{2}$ and $m + \frac{1}{2}$ existing between the distance 0 and ϱ, we get, substituting (2) (10) (11) in (1):

$$(N_m)_0^\rho = [G\,A + H\,B]\, \Delta(0) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which $\mu = mod.$ of the Nep-Log.

$$G = \frac{2\pi a}{h_m} e^{-\frac{0.4m - 2.20 + T}{\mu} + \frac{1}{4a^2\mu^2}} \Theta\left[a(2.20 - T - 0.4m + 2 log \varrho) - \frac{1}{2a\mu}\right] \quad (13)$$

$$H = \frac{2\pi a}{h_m} e^{-\frac{0.4m - 2.20 + T}{2\mu} + \frac{1}{16a^2\mu^2}} \Theta\left[a(2.20 - T - 0.4m + 2 log \varrho) - \frac{1}{4a\mu}\right] \quad (14)$$

and

$$log\ h_m = 2.20 - 0.4\, m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

As soon as the $(N_m)_0^\rho$ have become known we find the $(N_m)_{\rho_1}^{\rho_2}$ by simple subtraction.

I have carried through the solution for the values 400, 600, 800 and 1000 for g. It appeared that only when we come to the last value we get satisfactory results.

It seems superfluous to give all my calculations in full. I will only communicate some of the values obtained with the constant

$$g = 1000 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which was finally adopted. The value of the G and H were found to be as follows. (See table I p. 632).

Now, if for the stars of magnitude 2, 3, 4, 5, we take the numbers found by PICKERING for the whole of the sky, *viz.* resp. 58, 172, 577, 1848 [1]) and for the remaining magnitudes, the numbers which we derive from table 2 of the Groningen Publication N°. 18, by simply multiplying with 41 253 (the number of square degrees for the whole of the sky), we find equations of condition for the derivation of the unknown quantities $\Delta(0)$, D_{80}, D_{60}, such as this:

$$58 \frac{1}{\Delta(0)} = 0.2962 + 0\ 0411\, D_{80} + 0.0244\, D_{60}$$

etc. They get a more convenient form if we put $\frac{140}{\Delta} = Z$ and if we then divide all the equations by the coefficient of Z. In this way the equations of condition become as follows:

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[1]) In Publ. 18. p. 8 I found, by countings made on the materials of PICKERING:

$$N_{1.495}^{2.495} = 58;\ N_{2.495}^{3.495} = 171;\ N_{3.495}^{4.495} = 574;\ N_{4.495}^{5.495} = 1837.$$

With the aid of the computed values communicated in the same publication it is easy to pass from these to the numbers $N_{1.5}^{2.5}$ etc. The results thus found are those of the text.

TABLE I.

	G.				H.			
	$\rho = 10$	30	50	1000	$\rho = 10$	30	50	1000
$m = 2$	0.9341	2.365	3.105	4.834	0.2463	0 3302	0.3496	0 3670
3	3.406	10.75	15.39	30.29	0.7842	1 203	1.324	1.461
4	11.22	44.74	70.58	188.1	2.314	4.179	4.847	5.813
5	33.29	169.5	296.8	1 151.	6.275	13.66	16.94	23.12
7	210.9	1 795.	3 937.	39 430.	34.79	116.5	171.0	361.9
9	850.9	12 380.	34 570.	1 107 000.	128.0	696.0	1 254.	5 425.
11	2 163.	54 580.	195 700.	23 250 000.	304.6	2 778.	6 288.	72 510.
13	3 436.	151 700.	703 600.	340 600 000.	460.9	7 203.	20 780.	784 150.
15	3 387.	264 000.	1 589 000.	3 329 000 000.	438.6	11 900.	44 120.	6 275 000.

$$
\left.\begin{array}{llll}
(m=2) & 0.099\,D_{30} & +\,0.059\,D_{60} & -\,Z=0.715 \\
(m=3) & 0.186 & +\,0\ 146 & -\,Z=0.836 \\
(m=4) & 0.272 & +\,0.287 & -\,Z=0\ 817 \\
(m=5) & 0.375 & +\,0.534 & -\,Z=0.781 \\
(m=7) & 0.527 & +\,1.474 & -\,Z=0.595 \\
(m=9) & 0.508 & +\,3\ 108 & -\,Z=0.345 \\
(m=11) & 0.341 & +\,5.187 & -\,Z=0.149 \\
(m=13) & 0.159 & +\,6.926 & -\,Z=0.047 \\
(m=15) & 0.050 & +\,7.148 & -\,Z=0.010
\end{array}\right\} \quad . \quad . \quad (17)
$$

In solving these equations I have neglected those corresponding to the magnitudes 2 and 3. The reason is that for these the influence begins to be sensible of stars of so great a luminosity that *extrapolation* beyond the directly determined part of the luminosity-curve becomes necessary. These stars might therefore rather be used for a correction of this curve at its brighter extremity.

The remaining equations have been condensed into three by combining those for $m=4$ and 5, those for 7, 9, 11 and those for 13 and 15. The solutions of these three equations is:

$$
\left.\begin{array}{l}
Z=1.002 \quad \text{therefore } \Delta(0)=139.7 \\
D_{30}=0.460 \\
D_{60}=0.1315
\end{array}\right\} \quad . \quad . \quad . \quad . \quad (18)
$$

whereas we already assumed

$$
D_{10}=0\ 970 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)
$$

If with these values we compute the numbers N_m and if further we interpolate those for $m=6$, 8, 10, 12, 14 we get the following comparison between theory and observation:

TABLE II. TOTAL NUMBER OF STARS.

m	Obs.	Comp.	$O-C$ in fraction of whole obs. numb.
2	58	44.5	+ 0.233
3	172	161.5	+ 0.061
4	577	564.3	+ 0.022
5	1 848	1 889	— 0.022
6	5 816	6 025	— 0.036
7	17 940	18 450	— 0.028
8	54 040	54 580	— 0.010
9	159 200	157 200	+ 0.013
10	457 900	448 000	+ 0.022
11	1 275 000	1 256 000	+ 0.015
12	3 453 000	3 490 000	— 0.011
13	9 157 000	9 419 000	— 0.028[5]
14	23 680 000	24 100 000	— 0.018
15	60 225 000	58 500 000	+ 0.029

If, in accordance with what has been said, we except the very brightest magnitudes, the deviations are doubtlessly below the uncertainties in the determination of the observed numbers of the stars. The somewhat irregular course of the numbers is probably due to the discontinuities in the density-curve as definitively adopted.

The following table may serve to get at least some insight in the distribution of the stars of a determined apparent magnitude over the different distances.

TABLE III. NUMBER OF STARS $(N_m)_{\rho_1}^{\rho_2}$

ρ	$m=3$	5	7	9	11	13	15
0 to 10	108.	863	4770	17 500	42 000	63 000	60 000
10 » 30	45.5	779	8340	56 100	236 000	625 000	1 030 000
30 » 50	5.5	145	2350	3 400	144 000	543 000	1 250 000
50 » 1000	2.5	103	3000	59 900	835 00	8 188 000	56 150 000

or expressed in fractions of the totals

TABLE IV.

ρ	3	5	7	9	11	13	15
0 to 10	0.669	0.457	0.259	0 112	0.033	0.007	0.001
10 » 30	.282	.412	452	.357	.188	066	.018
30 » 50	.034	.077	.127	.150	.115	.058	.021
50 » 1000	015	.054	162	382	.665	.870	.960

Summing up we find: that the total numbers of stars of different magnitude (Harvard scale) as derived from observation in *Publications of the Groningen Laboratory No.* 18, are well represented by adopting the luminosity-curve (2), with the values (4) of the constants and the following values of the star-density:

TABLE V. STAR-DENSITY.

ρ	Δ	ρ	Δ
0	1.000$\Delta(0)$	100	0.125$\Delta(0)$
10	0 970	200	0.111
20	0 715	300	0.097
30	0 460	400	0.083
40	0.296	500	0.069
50	0.131^{5}	600	0 055
60	0.130	700	0.042
70	0 129	800	0.028
80	0.127	900	0 014
90	0.126	1000	0 000

in which $\Delta(0) = 139.7$.

It would be interesting to investigate what are the changes that can be made in these values without their ceasing to represent the observed numbers satisfactorily. I have deferred this investigation for the present because it will be desirable in such a discussion to include also the data furnished by the parallaxes.