Mathematics. — *"On fourdimensional nets and their sections by spaces."* (Second part). By Prof. P. H. Schoute.

(Communicated in the meeting of February 29, 1908).

The net (C_8).

1. The problem to determine the section of the net (C_8) with a given space can be naturally divided into two parts. The first part occupies itself with the question, how a series of spaces parallel to the given one intersects an eightcell; in the second is indicated, how the section of each of the eightcells intersected by the given space can be deduced from that section which determines this space in the eightcell assumed in the first part. Of course the four series of parallel spaces normal to an axis of the eightcell come here to the fore and then in the first part of the problem are investigated in the first place the so-called "transition forms" where the intersecting space contains one or more vertices of the eightcell, whilst between each pair of transition forms adjacent to each other a single intermediary form is introduced, namely that one by the space which bisects the distance between the two spaces bearing those transition forms. Generally this is sufficient for our end; moreover it is not difficult to interpolate where necessary other intermediary forms.

In the preceding communication of the same title we have packed up each of the cells C_{16} of the net (C_{16}) and each of the cells C_{24} of the net (C_{24}) in the smallest possible eightcell with edges parallel to the axes of coordinates, with the intention to connect the spacial sections of the nets (C_{16}) and (C_{24}) with those of the net (C_8) by cutting with each C_{16} and each C_{24} also the case C_8 enclosing these cells. With a view to this application we add to the above indicated four series of parallel intersecting spaces two others, viz. those normal to one of the two lines connecting the origin of coordinates with the point $(3, 1, 1, 1)$ and the point $(2, 1, 1, 0)$; indeed, these lines are — see the last table of the preceding communication — axes of one or more of the cells C_{16} and C_{24} enclosed in a cell C_8. Also for these two new series we restrict ourselves to the forms of transition and the intermediate forms lying in the middle between two adjacent forms of transition.

In order to simplify the survey of the sections appearing in the six series of parallel spaces we give the results to which the first part — the determination of the section with one C_8 — leads in two different ways. In the first place we project all vertices, edges, faces, bounding bodies of the cell C_8 on the axis normal to each of the six series of spaces to deduce the sections from this tabularly; in the second place we indicate the sections themselves in parallel perspective in the eightcell. To each of those two closely allied modes of transacting an extending plate is given.

To promote the uniformity we indicate the axes OE, OK, OF, OR by their ends $(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$. Then we have to deal successively with the six series

$(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$, $(3, 1, 1, 1)$, $(2, 1, 1, 0)$

and we have now to investigate for each of those six cases the two parts into which the problem was above divided.

2. *Case* $(1, 1, 1, 1)$. — This case was, as far as the first part of the problem is concerned, completely solved in a foregoing study (*Proceedings*, Jan. 1908, page 485). Hence the first part of the first plate with the superscription $(1, 1, 1, 1)$ OE_8 is an extension of the first diagram $n = 4$ of the plate given then. In order to be able to indicate together with the projections of all bounding elements the projections of the vertices of these elements, which considerably promotes the insight into the spacial figure, the numbers of edges, faces, bounding bodies are denoted here outside the scheme on the righthand side. Moreover the sections of the eightcell with the spaces of transition and the intermediate spaces perpendicular to the diagonal of projection are mentioned tabularly; here use has been made of a method formerly (*Verhandelingen*, volume IX, n°. 4) developed in all details which acquaints us not only with the characteristic numbers (e, k, f) of each section, but also with the nature of the faces. Thus the central section is a (6, 12, 8), because it contains 6 vertices and does not cut an edge, intersects 12 faces and contains no edges, intersects 8 bounding cubes and contains no faces; this section is a regular octahedron in connection with which each cube of the two quadruples of bounding bodies is cut according to an equilateral triangle of the same size. In this way the adjacent intermediary section is a (12, 18, 8), because 12 edges, 18 faces and 8 bounding cubes are intersected, viz. a tetrahedron regularly truncated at the vertices, i. e. the first of the semi-regular Archimedian polyhedra (*Proceedings*, page 488) because four of the bounding cubes are intersected according to regular hexagons, the four remaining ones according to equilateral triangles. Here the number of edges is found back as half of the total number of sides of the faces, thus 12 as half the product of eight and three, 18 as half the sum of four times six and four times three. Moreover, when indicating the polygons lying in the faces, we have underlined the figure of each group of regular polygons.

The second plate indicates the obtained sections in parellel perspective. The first diagram on the top leftside, represents an eightcell which indicates besides the diameters normal to the different series of parallel intersecting spaces a few other lines appearing in the solution; for our case (1, 1, 1, 1) to which the four following diagrams refer the axis EE' is this diameter. To characterize this case the mark $(1, 1, 1, 1)$ is noted down to the right at the bottom in the rectangle; moreover the fractions $\frac{4}{8}, \frac{3}{8}, \frac{2}{8}, \frac{1}{8}$ placed to the left at the top of each diagram indicate the part of the axis EE' lying with E on the same side of the intersecting space. It is easy to follow in these diagrams the changes in form which each face of the regular octahedron forming the central section undergoes, when the point of intersection of the intersecting space with the axis OE moves from O to E. Thus the face lying in the upper cube of the eightcell, which is at the same time the visible upper plane of the octahedron regarded by itself, transforms itself first into a regular hexagon, then into an equilateral triangle of opposite orientation, etc.; if the eightcell is a $C_8^{(2)}$, then the sides of the triangles of the first and third diagrams are $2\sqrt{2}$, those of the hexagons and the triangles of the second and fourth diagrams are $\sqrt{2}$, whilst the series closes with the transition form consisting of the single vertex E to which the fraction $\frac{0}{8}$ answers.

We now arrive at the question how the remaining eightcells that are likewise cut by the intersecting space are intersected in each of the considered cases. To this end we suppose the above intersected eightcell to be the central one of the net and so we assume the centre of this cell to be the origin of the system of coordinates with respect to which we have determined in the first communication the coordinates of the centres of the remaining cells in the symbolic form $(2a_i)$. The equation of the central space perpendicular to the axis OE_8 towards the point $(1, 1, 1, 1)$ is $x_1 + x_2 + x_3 + x_4 = 0$; the length of the normal let down out of the centre $(2a_i)$ on to this space is therefore Σa_i. So the eightcell with the centre $(2a_i)$ is cut by the central space $\Sigma x_i = 0$, when $-2 \leqq \Sigma a_i \leqq 2$, and here the five cases occur where Σa_i has one of the values $-2, -1, 0, 1, 2$. In other words: if with the central cell the central section $\frac{4}{8}$ makes its appearance, then with the remaining cells the sections $\frac{0}{8}, \frac{2}{8}, \frac{4}{8}, \frac{6}{8}, \frac{8}{8}$ occur and no others. The sections $\frac{0}{8}$ and $\frac{8}{8}$ being points and therefore not under consideration, we find as section of the net (C_8) a threedimensional space-filling consisting of two groundforms, octahedron and tetrahedron, where the tetrahedron occurs in two positions of opposite orientation. From a close consideration of this result follows now that the fractional symbols of the intersected cells furnish *in general* differences of multiples of quarters with that of the central cell and are thus represented by $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$ when the symbol of the central cell is $\frac{3}{8}$ or $\frac{1}{8}$. We find then again a threedimensional space-filling consisting of two groundforms each of which appearing in two oppositely orientated positions, the first semi-regular Archimedian body and the tetrahedron. As we arrive again at eightcell and tetrahedron when starting from the section $\frac{2}{8}$ of the central cell, the above-mentioned two cases are for this series the only ones where the threedimensional space-filling consists of two groundforms. In every other case — as e. g. the one answering to the fractions $\frac{1}{16}, \frac{5}{16}, \frac{9}{16}, \frac{13}{16}$ — we find four different groundforms and never more; we recommend the designing of the just mentioned quadruplet of sections as a good practice.

If we exchange the infinite system of cells $C_8^{(2)}$ by a finite block of k^4 cells $C_8^{(2)}$ forming together a $C_8^{(2k)}$, if we divide a diagonal of this block into eight equal parts and if we suppose the block to be intersected by a space standing in one of the points of division perpendicular to the diagonal, we then find according to circumstances either a finite system of octahedra $O^{(2\sqrt{2})}$ and tetrahedra $T^{(2\sqrt{2})}$ with edges $2\sqrt{2}$, or a finite system of Archimedian bodies $A^{(\sqrt{2})}$ and tetrahedra $T^{(\sqrt{2})}$ with edges $\sqrt{2}$, enclosed in an octahedron, a tetrahedron or an Archimedian body of greater size, viz., in the section of the block $C_8^{(2k)}$ with the intersecting space. In connection with the notes joined to the pages 15, 16 and 24 of the study "On the sections of a block of eightcells, etc." (*Verhandelingen*, volume IX, n°. 7) we here indicate how large in each of those cases the number of the component parts $O^{(2\sqrt{2})}$, $A^{(\sqrt{2})}$, $T^{(2\sqrt{2})}$, $T^{(\sqrt{2})}$ is. We restrict ourselves here to mentioning the results and we only remind the readers that the deduction of these are based on the actual connection

$\frac{1}{4}\,C_8^{(2k)} = T^{(2k\sqrt{2})}$	$T_p^{(2\sqrt{2})} \ldots\ldots (k+2)_3$ $T_n^{(2\sqrt{2})} \ldots\ldots (k)_3$	$O^{(2\sqrt{2})} \ldots\ldots (k+1)_3$
	Sum $\frac{1}{2}k(k^2+1)$	
$\frac{3}{8}\,C_8^{(4k)} = A^{(2k\sqrt{2})}$	$T_p^{(2\sqrt{2})} \ldots \frac{1}{6}k(23k^2+6k-2)$ $T_n^{(2\sqrt{2})} \ldots \frac{1}{6}k(23k^2-6k-2)$	$O^{(2\sqrt{2})} \ldots \frac{1}{6}k(23k^2-1)$
	Sum $\frac{1}{6}k(69k^2-5)$	
$\frac{1}{2}\,C_8^{(2k)} = O^{(2k\sqrt{2})}$	$T_p^{(2\sqrt{2})} \ldots 4(k+1)_3$ $T_n^{(2\sqrt{2})} \ldots 4(k+1)_3$	$O^{(2\sqrt{2})} \ldots \frac{1}{3}k(2k^2+1)$
	Sum $k(2k^2-1)$	
$\frac{1}{8}\,C_8^{(4k+2)} = T^{(2k+1)\sqrt{2}}$	$T_p^{(\sqrt{2})} \ldots (k+3)_3$ $T_n^{(\sqrt{2})} \ldots (k)_3$	$A_p^{(\sqrt{2})} \ldots (k+2)_3$ $A_n^{(\sqrt{2})} \ldots (k+1)_3$
	Sum $\frac{1}{3}(2k^3+3k^2+7k+3)$	
$\frac{3}{8}\,C_8^{(4k+2)} = A^{(2k+1)\sqrt{2}}$	$T_p^{(\sqrt{2})} \ldots \frac{1}{6}k(k+1)(23k+34)$ $T_n^{(\sqrt{2})} \ldots \frac{1}{6}k(23k^2+12k-11)$	$A_p^{(\sqrt{2})} \ldots \frac{1}{6}(k+1)(23k^2+19k+6)$ $A_n^{(\sqrt{2})} \ldots \frac{1}{6}k(23^2+27k+10)$
	Sum $\frac{1}{3}(46k^3+69k^2+29k+3)$	

between the coefficients of the different powers of x in the development of $(1+x+x^2+\ldots+x^{k-1})^4$ and the numbers of cells $C_8^{(2)}$ of the block $C_8^{(2k)}$ which agree with each other in projection on a diagonal. In the following table of results we have separated from one another the three cases leading to sections $\frac{1}{4}\,C_8^{(2)}=T^{(2\sqrt{2})}$, $\frac{1}{2}\,C_8^{(2)}=O^{(2\sqrt{2})}$ and the two cases leading to sections $\frac{1}{8}\,C_8^{(2)}=T^{(\sqrt{2})}$, $\frac{3}{8}\,C_8^{2}=A^{(\sqrt{2})}$. Moreover, the two positions of opposite orientation appearing for T and A are distinguished from each other as T_p, T_n and A_p, A_n, and then those parts $T^{(\sqrt{2})}$ and $A^{(\sqrt{2})}$ get the same foot-index which answer not only as regards volume but also as regards position of juncture to the relation

$$A^{(\sqrt{2})}+4\,T^{(\sqrt{2})}=T^{(3\sqrt{2})}\,,$$

whilst this index is a p (positive) for $T^{(2\sqrt{2})}$ when this tetrahedron agrees in position to $T^{(2k\sqrt{2})}$ and $A^{(2k\sqrt{2})}$, and can be taken arbitrarily in the third case $O^{(2k+2)}$, where the two amounts are indeed equal.

In this table the symbols $(k+2)_3$, etc. represent binomial coefficients. The coming to the fore of the numerical factor 23 is connected with the relation holding only for the volume

$$A^{(\sqrt{2})}=23\,T^{(\sqrt{2})},$$

which ensues immediately from the one given above. It forms part of

$$\frac{O^{(2\sqrt{2})}}{32}=\frac{A^{(\sqrt{2})}}{23}=\frac{T^{(2\sqrt{2})}}{8}=\frac{T^{(\sqrt{2})}}{1},$$

of which we have availed ourselves when arranging the preceding table, either as an aid in the calculation or as control.

The cases (1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0). — These three cases are so much simpler than the preceding one, that we can treat them collectively, now that the application of the results appearing here to the nets (C_{16}) and (C_{24}) make a short treatment necessary. The projection of the bounding elements on the corresponding axes OK, OF, OR are immediately found; in order to take into account the duality, appearing on one hand between OE and OR and on the other hand between OK and OF, the projections on OR are placed on the first plate next to those on OE, whilst the projections on OK and OF find a place there side by side. A single glance given to these diagrams already arouses the conviction that the sections in the direction of DE over OK and OF to OR must keep on becoming simpler. That this is really the case — and for what reason — is clearly evident from the second plate, giving the sections for the cases OK and OF. As is shown in the three diagrams with the fractional symbols $\frac{3}{6}, \frac{2}{6}, \frac{1}{6}$ belonging to OK here one of the dimensions of the section, viz. the dimension in the direction of the edge with K as centre, is of constant length, by which the sections become prisms with a height 2, namely an hexagonal prism $H^{(\sqrt{2})}$, a triangular prism $P^{(2\sqrt{2})}$ and a triangular prism $P^{(\sqrt{2})}$; with these symbols H and P the indices $\sqrt{2}$ and $2\sqrt{2}$ indicate the length of the sides of the bases. As a matter of fact we can now assert that with these prisms of which the endplanes are the determining variable elements, the problem of the intersection has lost a dimension; for, in order to determine the prism we have only to ask how the groundcube is intersected by a plane perpendicular to a diagonal of this bounding body of the eightcell, i. o. w. the problem has become threedimensional. In the same way we find in case OF rectangular prisms of which two dimensions remain constant, which has been indicated for the section of transition $\frac{2}{4}$ and the intermediary section $\frac{1}{4}$, whilst the section in case OR is an invariable cube, which is of course not designed.

It is almost superfluous to stop for the two space-fillings of case OK, that by $H^{(\sqrt{2})}$ and $P^{(\sqrt{2})}$ together and that by $P^{(2\sqrt{2})}$ alone, as they appear indeed as well-known plane-fillings. We suffice by giving the following relations:

$$\left.\begin{array}{lllll}
P^{(2k+1)\sqrt{2}} = & (k+2)_2\, P^{(\sqrt{2})}_p + & (k+1)_2\, H^{(\sqrt{2})} + & (k)_2\, P^{(\sqrt{2})}_n \\
H^{(2k+1)\sqrt{2}} = & 6(k+1)_2\, P^{(\sqrt{2})}_p + & (3k^2+3k+1)\, H^{(\sqrt{2})} + & 6(k+1)_2\, P^{(\sqrt{2})}_n \\
P^{(k\sqrt{2})} = & (k+1)_2\, P^{(\sqrt{2})}_p & + & (k)_2\, P^{(\sqrt{2})}_n \\
H^{(k\sqrt{2})} = & 3k^2\, P^{(\sqrt{2})}_p & + & 3k^2\, P^{(\sqrt{2})}_n
\end{array}\right\}$$

4. *Case* (3, 1, 1, 1). — If the vertex A of the eightcell $C_8^{(2)}$ — see first diagram of second plate — is point $(1, 1, 1, 1)$ then the point $\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is obtained by dividing the inner diagonal AB of the cube lying in the space $x_1 = 1$ into three equal parts and then to take the first point of division C[1]). The line OC is for this case

[1]) By mistake in the diagram for AC has been taken $\frac{1}{3} AR$ instead of $\frac{1}{3} AB$.

the axis upon which we must project to determine the projec of the bounding elements. Now it is clear that the projection of cube with AB as a diagonal is obtained by projecting first bounding body on the projection AB of the axis OC on its sp $x_1 = 1$ which furnishes with regard to the vertices the stratifica 1, 3, 3, 1 and by determining then the projection on OC of th new points lying on AB. Now, angle BOC is a right one, for of the coordinates $(1, -1, -1, -1)$ and $\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ of B C follows immediately $OB^2 + OC^2 = BC^2$. So B projects itself OC in O and so this of course is also the case with the ve $(-1, 1, 1, 1)$ of the eightcell lying opposite B. So we find — the first plate under head (3, 1, 1, 1) C_8 — the stratification the 16 vertices by causing the group of points 1, 3, 3, 1 upon the axis of projection at equal intervals to be followed a second group of the same structure in such a way that the 1 of this second group coincides with the last 1 of the first grc It is from this that this projection has its type, as is indicatec the foot. One really finds without any difficulty all that is giver the scheme by representing to oneself the two bounding cubes i cated in the typical image — here lying in the spaces $x_1 = \pm 1$ and to suppose that their corresponding vertices, edges, faces united by edges, faces and bounding bodies.

If again we do not take the isolated point A into considerat then we have to deal here with six different forms of the section, th intermediary forms and three forms of transition; these are given v the addition of the corresponding fractional symbols $\frac{6}{12}, \frac{5}{12}, \ldots,$ on the second plate. We shall indicate somewhat in details h these diagrams are deduced by drawing, independently of the res of the first plate, and to this end we immediately notice that the sp through A perpendicular to OC is represented by $3x_1 + x_2 + x_3 + x_4 =$ and that this space after a slight parallel displacement to O trunca from the edges of the eightcell passing through A segments wh are in the ratio to each other of $1:3:3:3$. If now the edge A drawn horizontally is parallel to OX_1, we begin to set off, in or to obtain the first intermediary form, on the other edges throu A — see the last of the six diagrams — segments AP_2, AP_3, A to the length of half the edge, i. e. of the unit, on the edge A a segment AP_1 with a length of a third of the unit, which cau the tetrahedron $P_1P_2P_3P_4$ corresponding to the symbol $\frac{1}{12}$ to be ge

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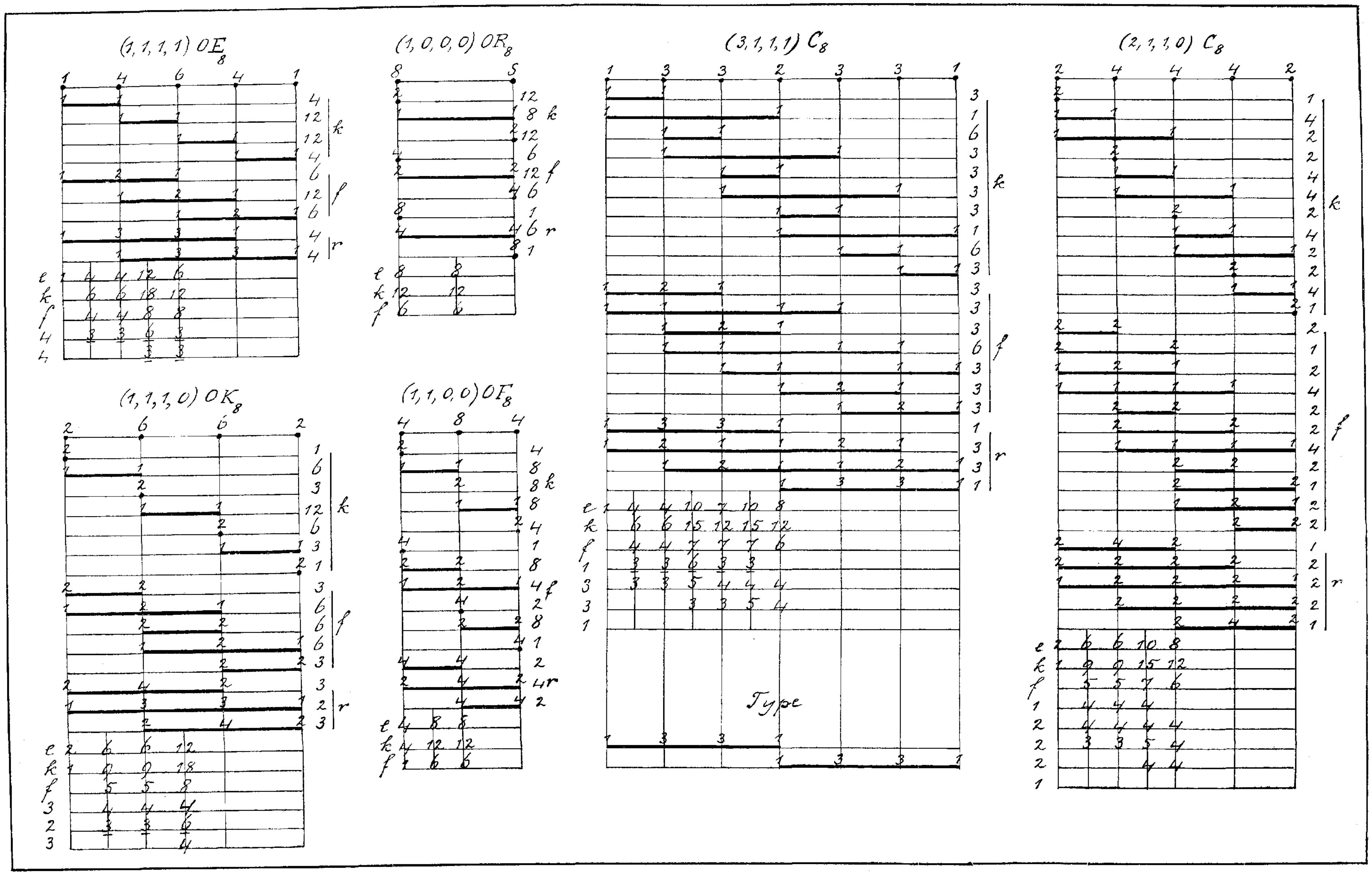
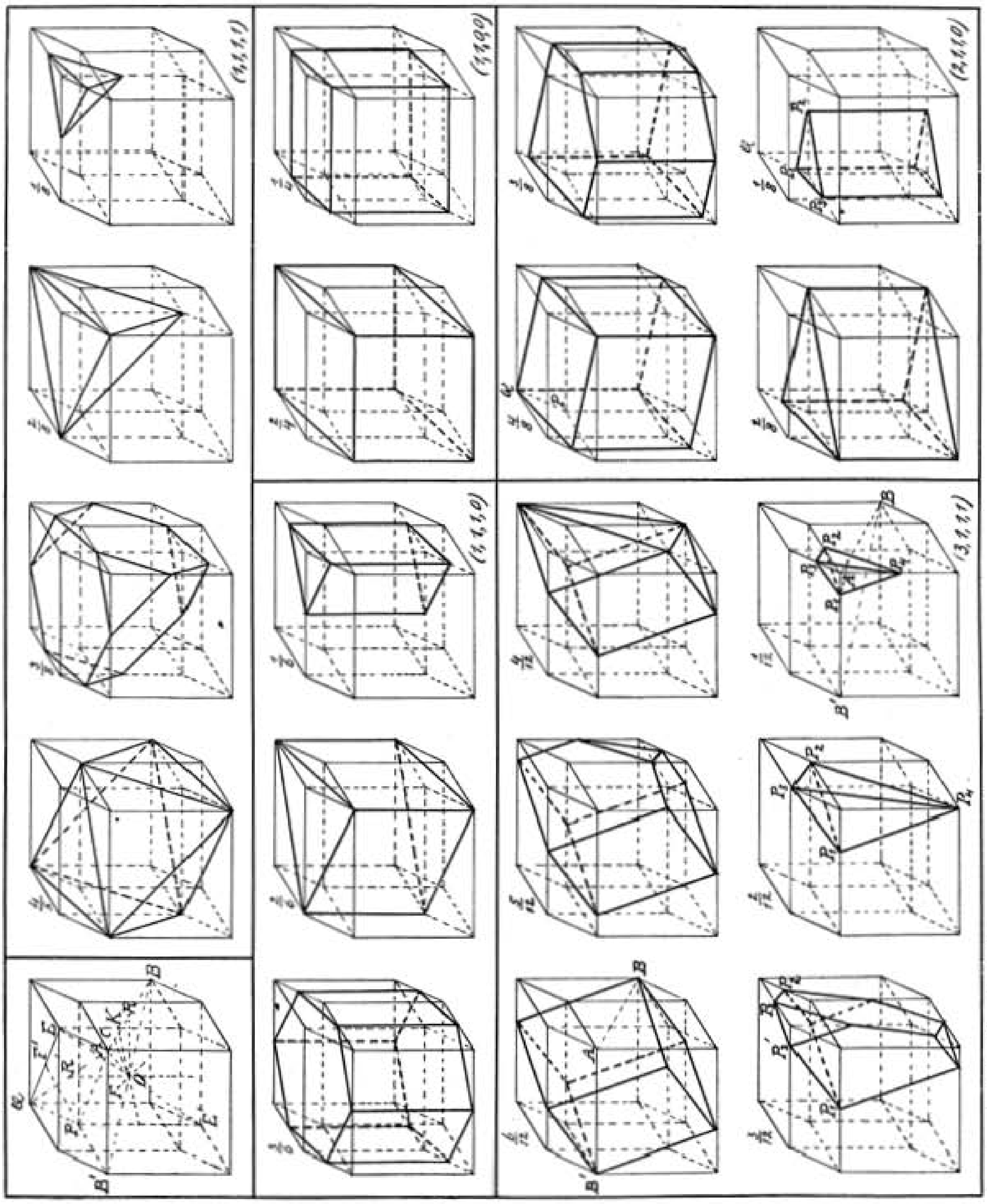
rated. The space $x_1 = 1$ contains of this tetrahedron the equilateral triangle $P_2P_3P_4$ with the side $\sqrt{2}$; the other faces $P_1P_3P_4$, $P_1P_2P_4$, $P_1P_2P_3$ lying in the spaces $x_2 = 1, x_3 = 1, x_4 = 1$ are isosceles triangles with basis $\sqrt{2}$ and sides $\frac{1}{3}\sqrt{10}$. So this section is not a regular *tetrahedron* but a regular *triangular pyramid*, of which the perpendicular let down out of the vertex P_1 on to the groundplane $P_2P_3P_4$ is an axis with the period three; because the foot of this perpendicular lies on the diagonal AB of the right cube at a distance from A forming a sixth part of AB and as AP_1 is likewise a sixth of AB' this axis is parallel to the diagonal BB' of the eightcell. It is now easy to deduce the changes of the section following from the displacement of the intersecting space by investigating either the parallel displacement of the edges of the section over the faces of the eightcell or the parallel displacement of the faces of the section through the bounding cubes of the eightcell. If the intersecting space has removed itself as far as double the distance from A, then — as is evident from both considerations — the tetrahedron of intersection has simply been multiplied by two from A. Passing on from this section $\frac{2}{12}$ it seems preferable to watch more closely the edges. If the edges P_3P_2 and P_3P_1 of the section $\frac{2}{12}$ have arrived in the positions P_3P_2 and $P_3'P_1$ of the section $\frac{3}{12}$ when the intersecting space has come at the threefold distance from the starting point A, it is sufficiently evident that the connection of the points P_3P_3' must furnish a new edge. So we see gradually how the entire rhombohedron forming the section $\frac{6}{12}$ develops itself. We yet point to the fact that the section in each position of the intersecting space during its parallel motion has an axis with period three, parallel to the diagonal BB' and at last passing into this line. Indeed, the diagonal AB of the bounding cube lying in space $x_1 = 1$ being an axis of revolution with the period three for that cube, so the plane through AB and AB' is a "plane of revolution" with the period three for the eightcell. As now the moving intersecting space is and remains normal to the line OC lying in this plane — see the first of the 20 diagrams — the line of intersection of this plane with the intersecting space, which line is of course normal to OC, must be an axis with the period three for the section. As was found already above the line OB is really normal to OC and so the obt axis is parallel to OB. Because the plane through AB and B' con the perpendiculars OC and OR out of O on to the interse space and the space $x_1 = 1$ of the righthand cube, each line of i therefore also OB must be normal to the plane determined by intersecting space in the space $x_1 = 1$; so if we move the interse space in an opposite sense and return from $\frac{6}{12}$ by $\frac{5}{12}$, etc. to $\frac{1}{1}$ rhombohedron forming the central section, and then moving in direction of the edge $B'A$ through the eightcell, is truncated nc to the axis by the plane determined in the space of that right In fact, in the above mentioned paper (*Verhandelingen*, vol. IX, has been found that the section is always a rhombohedron or a trun rhombohedron when the intersecting space is normal to a through two opposite edges, which is here the case, as the through AB and B' contains the edge AB' and the opposite

We now indicate the body corresponding to the fractional sy $\frac{n}{12}$ by D_n, where n can take one of the values $1, 2, \ldots, 11, 12$ D_n and D_{12-n} represent the two oppositely orientated positions selfsame body, with a view to then investigating which of those make their appearance when the net (C_8) is cut by the central $3x_1 + x_2 + x_3 + x_4 = 0$. From the distances of the points witl coordinates $(2a_i)$, forming the system of centres of the net, fo immediately that the parts $D_2, D_4, D_6, D_8, D_{10}$ appear together that thus the corresponding threedimensional space-filling co of three — and if we notice the orientation even of five — diff groundforms. Now, as we know, the form D_6 alone already is to fill the space and so this is also the case with the forms D D_8 and the forms D_4 and D_{10} together. What is more, from condition that in the obtained space-filling with the three or different groundforms the face of one of those forms must con itself in faces of the surrounding forms, follows immediately beside each D_2 must lie a completing D_8, beside each D_4 a compl D_{10} and that recomposition of those parts completing each othe D_6 must lead to a net of rhombohedra D_6. We really cause th of rhombohedra to be generated in a simpler way if, before cu the net (C_8) by the assumed space, we suppose the series o spaces $x_1 = 2a_1 + 1$ to have disappeared, a thing to which th of the plane of projection through the two edges, here AB' an opposite one, has led us involuntarily in the paper quoted last this the net (C_8) transforms itself into a threefold infinite net


(1, 1, 1, 1) OE_8
(1, 0, 0, 0) OR_8
(3, 1, 1, 1) C_8
(2, 1, 1, 0) C_8
(1, 1, 1, 0) OK_8
(1, 1, 0, 0) OF_8
Type



infinite series of rectangular prisms which have a cube with the edge two as basis, and the section of this net of prisms is exactly the net of rhombohedra. That the sections which, when the intersecting space has an arbitrary position, are quite irregular parallelopipeda, here become rhombohedra is the result of the fact that the intersecting space forms with each of the three spaces $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ equal angles, angles with a cosine of the value $\frac{1}{6}\sqrt{3}$. Out of the diagram with the symbol $\frac{6}{12}$ it is furthermore evident that the ends BB' of the axis of this rhombohedron lie in two consecutive spaces $x_1 = 2a_1 + 1$ and that the distance of the parallel spaces of intersection of the intersecting space with these spaces, which spaces cut the net of rhombohedra in the intersecting space into pieces, must amount to 4. This tallies; for the angle between the spaces $3x_1 + x_2 + x_3 + x_4 = 0$ and $x_1 = 0$ has $\frac{1}{2}\sqrt{3}$ as cosine and therefore $\frac{1}{2}$ as sine, so that the distance of the planes must be $2 : \frac{1}{2}$.

From the preceding follows now likewise that the section with the space $3x_1 + x_2 + x_3 + x_4 = 1$ furnishes a space-filling consisting of the parts D_1, D_3, D_5, D_7, D_9, D_{11}; of course also this space-filling consisting of three groundforms each of which appearing in two opposite positions can be obtained by cutting up a net of rhombohedra. It is also clear that by taking an intermediary position of the space of intersection we are led to six quite different groundforms, which can be indicated by $D_{\frac{1}{2}}$, $D_{2\frac{1}{2}}$ . . . . . $D_{10\frac{1}{2}}$, or in opposite orientation by $D_{1\frac{1}{2}}$, $D_{3\frac{1}{2}}$ . . . . $D_{11\frac{1}{2}}$.

By cutting a block of k^4 cells C_8 instead of a fourfold infinite net (C_8) we can also deduce how one of the forms $D_n^{(k)}$ of k-times greater linear size can be built up out of the above mentioned segments D_n. We avoid this not to become too longwinded.

5. *Case* (2, 1, 1, 0). — When treating the case (1, 1, 1, 0) we have seen that the appearance of nought in the symbol causes prisms to be found with the constant height 2, by which the fourdimensional problem is reduced to a threedimensional one. Thus we are placed before the consideration of the section (2, 1, 1) of the net of cubes which in various respects for the threedimensional space forms the analogon of that of the section (3, 1, 1, 1) in Sp_4.

If we suppose that the space, in which the section (2, 1, 1) is to be taken, contains the upper cube of the eightcell and the vertex P — see the first of the 20 diagrams — is taken as origin of a rectangular system of coordinates with the edges passing through this point as axes, the edge PQ as axis corresponding to the figure 2 of $(2, 1, 1)$, then the centre F' of the upper plane of that cube is the point $(2, 1, 1)$ and PF' is therefore the axis normal to the series of intersecting planes [1]). Now it follows from the rectangle $APQE$ with the sides $AE = 2$, $AP = 2\sqrt{2}$, that AQ is normal to PF' and that the points A and Q project themselves on PF' in the same point. Thus we find the projection of the eight vertices of the cube under consideration on PF' by placing the projections (1, 2, 1) of the faces with PA and QE as diagonals so side by side that the last 1 of the first coincides with the first 1 of the last, by which the stratification 1, 2, 2, 2, 1 is arrived at, which, with a view to upper and lower cube, passes by doubling into 2, 4, 4, 4, 2. From this ensue then the results given on the first plate. If we now — returning to the second plate — set off on the three edges of the cube passing through P, in the assumed supposition that PQ agrees with the 2 of $(2, 1, 1)$, from P segments $\frac{1}{2}$, 1, 1 then — see the last diagram — the triangle $P_1P_2P_3$ appears forming the upper plane of the triangular prism corresponding to the fraction $\frac{1}{8}$ and out of this the sections $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$ are developed in the same way as was pointed out above. Of triangle $P_1P_2P_3$ the line connecting P_1 with the middle of P_2P_3 is an axis with the period two, or to express it more simply a line of symmetry, and this line is parallel to the diagonal AQ of the first diagram. In each position of the intersecting plane the section has the line of intersection of this plane with the plane $APQE$ as line of symmetry; in connection with this the lozenge, unmutilated for the case $\frac{4}{8}$, which when following the reverse way to the case $\frac{1}{8}$ moves parallel to itself through the cube in such a way that the vertex Q describes the edge QP, is cut by the groundplane of the cube according to a perpendicular on the line of symmetry. If we imagine in the chosen space of the upper cube of the eightcell the threefold net of cubes and if we remove before passing to the intersection by the series of parallel planes the partitions parallel to the endplanes, we obtain in the intersecting plane a net of lozenges which are cut by the removed partitions into segments of the found form, etc.

In the ensuing parts we shall pass on to the intersection of the nets (C_{16}) and (C_{24}).

1) It is really inaccurate to speak of an upper plane of the upper cube; of course the plane is meant, which appears in the diagram as upper plane to the eye.