Mathematics. — "*On the cyclic minimal surface*". By Prof. J. C. KLUYVER.

(Communicated in the meeting of January 25, 1908).

ENNEPER (Zeitschr Math. Phys. 14) pointed to the existence of a minimal surface containing a system of circles lying in parallel planes, with centres situated on a plane curve. Let us suppose that this curve passes through the origin of the rectangular coordinates, that it is situated in the XZ-plane and that the variable circle with the centre $(\xi, 0, \zeta)$ and the radius R, generating the surface, lies always in a plane parallel to the XY-plane.

The rectangular coordinates x, y, z of a point of the surface are given by the equations

$$x = \xi + R\cos\alpha \quad , \quad y = R\sin\alpha \quad , \quad z = \zeta,$$

so that they are expressed in the two parameters α and ζ. We find that the differential equation of the minimal surfaces is satisfied when

$$R^2(\xi'' R\cos\alpha + RR'') - R^2(1 + \xi'^2 + R'^2 + R'^2 + 2\xi' R'\cos\alpha) = 0,$$

in which equation the dashes denote the differentiations with regard to ζ.

The equation breaks up into

$$\xi'' R = 2\xi' R'$$

and into

$$RR'' = 1 + \xi'^2 + R'^2.$$

The first equation furnishes

$$\xi' = \frac{AR^2}{b^2},$$

where A denotes a positive constant and b the minimum value of R.

The second equation now passes into

$$\frac{d}{d\zeta}\left(\frac{R'}{R}\right) = \frac{1}{R^2} + \frac{A^2R^2}{b^4}$$

and the integration furnishes

$$R'^2 = \frac{1}{b^2}(R^2 - b^2)\left(1 + \frac{A^2R^2}{b^2}\right),$$

so that finally we can express ξ en ζ in R by means of elliptic integrals.

We find

$$\xi = \frac{A}{B}\int_b^R \frac{dR}{\sqrt{(R^2-b^2)\left(1+\frac{A^2R^2}{b^2}\right)}} \quad , \quad \zeta = b\int_b^R \frac{R^2dR}{\sqrt{(R^2-b^2)\left(1+\frac{A^2R^2}{b^2}\right)}}.$$

Here an elliptic argument can be introduced. We put

$$R = \frac{b}{cn\, u},$$

$$k = \sin\theta = \frac{1}{\sqrt{1+A^2}},$$

and we find

$$\xi = bk' \int_0^u \frac{dw}{cn^2 w} \quad , \quad \zeta = bku.$$

By allowing u to vary from $-K$ to $+K$ the centre M with the coordinates ξ, ζ in the XZ-plane describes completely the locus of the centres and the equation

$$R = \frac{b}{cn u}$$

indicates how the radius of the circle changes during the motion.

We notice that the minimal surface depends on two constants b and k, that the smallest circle ($u = 0$) is found in the XY-plane, that with respect to the origin there is symmetry, and that for $u = K$, $\zeta = bkK$ the radius R has become infinite whilst at the same time the centre M is at infinite distance.

As however

$$\lim_{u=K} (\xi - R) = b \lim_{u=K} \left[k' \int_0^u \frac{dw}{cn^2 w} - \frac{1}{cn\, u} \right] = \frac{b}{k'} (k'^2 K - E)$$

and $\xi - R$ retains therefore a finite value the surface contains two right lines

$$z = \pm\, bk\, K,$$

$$x = \pm \frac{b}{k'} (k'^2 K - E).$$

For $k = 1$ the eliptic integrals degenerate. We have

$$\xi = 0, \quad \zeta = bu, \quad R = b\, Ch\, u,$$

and the surface has passed into a catenoid. The smaller k is, the more the surface deviates from the catenoid and the more oblique it becomes. For, we find for the coefficient of direction of the tangent to the locus of the centres M:

$$\frac{d\zeta}{d\xi} = \frac{k\, cn^2\, u}{k'}$$

and the greatest value of this coefficient $k : k'$, which is arrived at in the origin, tends to zero when k tends to zero. The surface is then altogether in the XY-plane.

I shall now endeavour first to investigate in the following wh it is possible to bring through two equal circles placed in parall planes a cyclic minimal surface and then to calculate the part of tl minimal surface extended between those circles.

When for both circles the radius R is taken equal to 1, the centr $M(\xi, \zeta)$ and $M'(-\xi, -\zeta)$ are situated in the XZ-plane symmetrical with respect to the origin and their planes are parallel to the X plane, the question is whether the two equations:

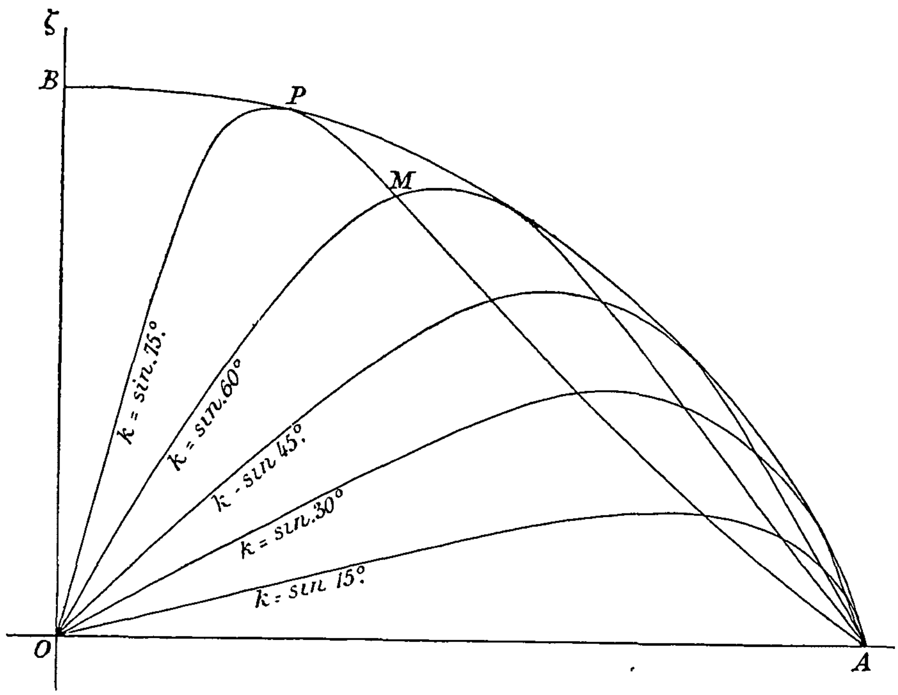
$$\xi = k' \, cn \, u \int_0^u \frac{dw}{cn^2 \, w}, \quad \zeta = ku \, cn \, u$$

admit of suitable solutions for k and u. If these are found, we ha $b = cn \, u$ and both parameters b and k of the minimal surface a known.

In order to investigate the indicated equations we regard for tl present in the $\xi\zeta$-plane ξ and ζ as variables and we consider tl curve which is described by point (ξ, ζ), when for constant k tl variable u describes the range of values from 0 to K. We have:

$$\xi(0) = 0 \quad , \quad \zeta(0) = 0,$$
$$\xi(K) = 1 \quad , \quad \zeta(K) = 0.$$

So for all values of k the curve will run from the origin O point A on the ξ-axis (see the diagram).



Farther we have:

$$\xi = k' \, cn\, u \int_0^u \frac{d\,(tn\, w)}{dn\, w} < k' \, cn\, u \int_0^u \frac{d\,(tn\, w)}{dn\, u},$$

$$\xi < \frac{k' \, sn\, u}{dn\, u},$$

so that from:

$$\frac{d\xi}{du} = \frac{1}{cn\, u}\,(k' - sn\, u\, dn\, u\, \xi)$$

follows:

$$\frac{d\xi}{du} > k' \, cn\, u.$$

We conclude that for increasing u the variable ξ grows regularly from 0 to 1. So the curve OA is intersected but once by a line $\xi =$ constant.

At the same time:

$$\frac{d\zeta}{du} = k\,(cn\, u - u\, sn\, u\, dn\, u) = k\, cn\, u \left(1 - u \frac{sn\, u}{sn\,(u+K)}\right).$$

For small u we find $\frac{d\zeta}{du}$ to be positive, it keeps on decreasing, becomes one time zero and is then negative. So the variable ζ reaches somewhere a maximum and the curve OA is either not cut by a line $\zeta =$ constant or in two points The form of the curve $k =$ constant is therefore as is indicated schematically in the diagram. In order to be able to compare the curves belonging to different values of k we can determine the values which the differentialquotient $\frac{d\zeta}{d\xi}$ assumes in the points O and A.

We have

$$\left(\frac{d\xi}{du}\right)_{u=0} = k' \;,\; \left(\frac{d\zeta}{du}\right)_{u=0} = k,$$

$$\left(\frac{d\xi}{du}\right)_{u=K} = E - k'^2 K \;,\; \left(\frac{d\zeta}{du}\right)_{u=K} = -\, kk'K,$$

from which ensues

$$\left(\frac{d\zeta}{d\xi}\right)_{\xi=0} = \frac{k}{k'} \;,\quad \left(\frac{d\zeta}{d\xi}\right)_{\xi=1} = -\frac{kk'K}{E - k'^2 K} = -\frac{k'K}{k \int_0^K cn^2 w\, dw}. \qquad (O)$$

From this is apparent that in O the value of $\frac{d\zeta}{d\xi}$ increases with k, that on the other hand the absolute value of $\frac{d\zeta}{d\xi}$ decreases in A for increasing k. For, if k becomes greater $k'K$ decreases, but the denominator $k\int_0^K cn^2\, w\, dw$ increases.

Taking into consideration the form just sketched of a curve OA belonging to a definite value of k we find that a second suchlike curve belonging either to a larger value or to a smaller value of k will certainly intersect the first curve somewhere. So as soon as a cyclic minimal surface passes through two equal circles placed in parallel planes we shall be able to bring a second cyclic minimal-surface through these circles.

We must now investigate when the two cyclic minimal surfaces coincide, i. o. w. we must find the envelope of the curves OA.

If we put $c = k^2$, $c' = k'^2$, then the system of curves is given in the equations

$$\xi = \sqrt{c'}\, cn\, u \int_0^u \frac{dw}{cn^2\, w} \quad , \quad \zeta = \sqrt{c}\, u\, cn\, u\ ;$$

we regard c as the parameter of the curve, $\varphi = am\, u$ as the parameter determining a point on a given curve, so that the coordinates (ξ , ζ) of a point of the envelope satisfy the condition

$$\frac{D(\xi , \zeta)}{D(c , \varphi)} = 0\ .$$

If here and in future we put for shortness' sake

$$A(u) = \int_0^u \frac{dw}{cn^2\, w} \quad , \quad B(u) = \int_0^u \frac{dw}{dn^2\, w}$$

and we take into account that for constant $\varphi = am\, u$ we have

$$\frac{\partial u}{\partial c} = \frac{1}{2c}\,(B(u) - u),$$

we find

$$\frac{\partial \xi}{\partial c} = -\frac{1}{2\sqrt{c'}}\, cn\, u\, B(u) \quad , \quad \frac{\partial \zeta}{\partial c} = \frac{1}{2\sqrt{c}}\, cn\, u\, B(u)\ ,$$

$$\frac{\partial \xi}{\partial \varphi} = \sqrt{c'}\, sn\, u\, (c\, B(u) - Q(u)) \ , \ \frac{\partial \zeta}{\partial \varphi} = -\sqrt{c}\, sn\, u\, (c'\, B(u) + Q(u))\ ,$$

where $Q(u)$ is given by the equations

$$Q(u) = u - E(u) - \frac{dn\, u\, cn\, u}{sn\, u}\ ,$$

$$= K - E - \int_u^K \frac{dw}{sn^2 w},$$

$$= \frac{1}{sn^2 u}\left(u - cn^2 u\, A(u) - dn^2 u\, B(u)\right),$$

$$= A(u) + k^2 B(u) - \frac{1}{sn\, u\, cn\, u\, dn\, u}.$$

From this ensues

$$\frac{D(\xi, \zeta)}{D(c, \varphi)} = -\frac{1}{2\sqrt{cc'}}\, cn\, u\, sn\, u\, B(u)\, Q(u),$$

and so the points of the envelope of the curves OA are determined by the equations

$$Q(u) = K - E - \int_u^K \frac{dw}{sn^2 w} = 0 \ ^{1)}.$$

As when c is given, the first member of the equation increases regularly from $-\infty$ for $u = 0$ to $K - E$ for $u = K$, the equation $Q(u) = 0$ admits of one solution u_0. By differentiating we find

$$\frac{du_0}{dc} = \frac{1}{2c}\int_0^{u_0} dw \left[\frac{dn^2 u_0}{dn^2 w} - 1\right],$$

i. e. a negative value; therefore the greater c is, the smaller is the argument u_0, which I call the critical argument. This argument moves finally between rather narrow limits. For $c = 0$ we find $K = E = \frac{\pi}{2}$ and so also $u_0 = \frac{\pi}{2} = 1.5708$. For $c = 1$ we find

$$Q(u) = u - E(u) - \frac{dn\, u\, cn\, u}{sn\, u} = u - \frac{1}{sn\, u} = u - \frac{Ch\, u}{Sh\, u}.$$

So the critical argument u_0 satisfies the equation

$$u_0 = \frac{Ch\, u_0}{Sh\, u_0}.$$

From this ensues

$$u_0 = 1.1997,$$

$$\varphi_0 = am\, u_0 = 56^\circ.28',$$

$$\cot \varphi_0 = u_0\, cn\, u_0 = 0.6627.$$

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[1]) G. JUGA. (Ueber die Constantenbestimmung bei einer cyklischen Minimalfläche, Math. Ann. Bd. 52) gives this equation in the form

$$cnu\, dnu + (E(u) - u)\, snu = 0.$$

For values of c between 0 and 1 it is easy to solve u_0 out of the equation

$$Q(u_0) = K - E - \int_{u_0}^{K} \frac{dw}{sn^2\, w} = 0$$

by means of the tables of LEGENDRE. If u'_0 is an approximate value of the critical argument, the calculation of NEWTON furnishes

$$u'_0 - Q(u'_0)\, sn^2\, u'_0$$

as following approximation. In this way the critical argument is calculated in the following table for some values of $k^2 = c$

$k = \sqrt{c}$	$\varphi_0 = am\, u_0$	u_0	$b = cn\, u_0$	ξ_0	ζ_0	φ'_0	ξ'_0	ζ'_0
sin 0°	90°	1.5708	0	1.	0.	90°	1.	0
15°	87° 1′	1.5442	0.0520	0.9966	0 0208	87° 0′	0.9954	0.0245
30°	79°17′	1.4701	0.1859	0.9498	0.1367	79°23′	0.9427	0 1423
45°	70° 3′	1.3708	0 3412	0.7930	0.3308	70°16′	0.7916	0.3325
60°	62°31′	1.2801	0.4614	0.5573	0.5116	62°35′	0 5549	0.5133
75°	57°57′	1.2198	0.5306	0 2813	0.6251	57°57′	0.2776	0.6265
90°	56°28′	1.1997	0.5524	0.	0.6627	56°28′	0.	0.6627

and moreover are indicated in it the coordinates ξ_0, ζ_0 of the point P, in which the curve OA belonging to each value of k touches the envelope of that system of curves.

By the equations

$$\xi_0 = \sqrt{c}\, cn\, u_0\, A(u_0) \quad , \quad \zeta_0 = \sqrt{c}\, x_0\, cn\, u_0$$

we now find in connection with the condition

$$Q(u_0) = 0$$

that ξ_0 and ζ_0 are given as functions of c only. We can deduce out of it

$$\frac{d\xi_0}{dc} = -\frac{1}{2\sqrt{c}}\, dn\, u_0\, B(u_0)\, [cn\, u_0\, dn\, u_0 + u_0\, c\, sn^3\, u_0],$$

$$\frac{d\zeta_0}{dc} = \frac{1}{2\sqrt{c}}\, dn\, u_0\, B(u_0)\, [cn\, u_0\, dn\, u_0 + u_0\, c\, sn^3\, u_0],$$

$$\frac{d\zeta_0}{d\xi_0} = -\frac{k'}{k}.$$

From this appears that for increasing k or $\sqrt{c}$ the coordinate ξ_0 decreases regularly and the coordinate ζ_0 increases regularly. In connection with the numbers inserted in the table it follows that the envelope of the curves OA has about the shape of a quadrant of ellipse BA of which half of the great axis $OA = 1$ and half of the small axis $OB = 0.6627$.

Moreover it is clear that the tangent to any curve $k =$ constant, in the point P where the latter touches the envelope, is normal to the tangent in the origin O drawn to this same curve. The preceding calculations now lead to the conclusion that through the two equal circles with radius $R = 1$ placed parallel and symmetrically with respect to the origin two cyclic minimal surfaces will pass, when the centre $M(\xi,\zeta)$ of the upper circle is situated inside the curve BA of the diagram, that the two surfaces coincide when M has arrived on the curve BA and that the circles cannot be connected by a minimal surface when M falls outside the curve BA.

If M lies inside the curve BA two curves OA pass through M. One of these touches the envelope in P, a point on curve OA between O and M. So the argument u belonging to M is greater than the critical argument u_0 in P and so the minimal surface belonging to it and extended between the circles M and M' would contain the two circles along which this minimal surface is cut by a second minimal surface with an infinitesimal slight difference. So this minimal surface is unstable. For the second minimal surface laid through the circles an argument u corresponds to M smaller than the critical argument u_0; this surface is therefore stable and can be realized in a proof of PLATEAU.

If two surfaces can be laid through the circles the most oblique surface (the surface belonging to the smaller value of k and with the greater value of the radius b of the mean section) is therefore always stable, the other is unstable.

It is worth mentioning that whilst here the quantities φ_0, ξ_0, ζ depend in rather an intricate way on $k = \sin\theta$, we can find by approximation out of simple formulae very accurate values for these quantities.

If we call the critital amplitude 56°28′ of the catenoid β, we shall be able to assume with great accuracy the following relations:

$$\cos\varphi_0 = \cos\beta \sin^2\theta \left(1 + \frac{4}{9}\cos^2\theta\right),$$

$$\xi_0^2 = 1 - \left(\frac{\cos\varphi_0}{\cos\beta}\right)^2,$$

$$\zeta_0 = \cot\beta \left(\frac{\cos\varphi_0}{\cos\beta}\right)^{\frac{7}{5}},$$

from which ensues for the equation of the envelope BA

$$\zeta_0^2 + \left(\frac{\zeta_0}{cot\,\beta}\right)^{\frac{10}{7}} = 1.$$

In the table the values of φ_0, ξ_0 and ζ_0 calculated in this way are added in the three last columns, to be compared.

To conclude with we give a computation of a part of a given cyclic minimal surface with given parameters b and k, situated beween two equally large circles corresponding to the arguments $+u$ and $-u$.

The coordinates x, y, z, of a point of the surface are again determined by the equations:

$$x = bk' A(u) + \frac{b}{cn\,u} cos\,\alpha, \quad y = \frac{b}{cn\,u} sin\,\alpha, \quad z = bk\,u,$$

out of which we can find for the line-element on the surface the expression

$$\frac{ds^2}{b^2} = \frac{Pdu - i\,cn\,u\,d\alpha + i\,k' \,sin\,\alpha\,du}{cn^2\,u} \times \frac{Pdu + i\,cn\,u\,d\alpha - i\,k'\,sin\,\alpha\,du}{cn^2\,u},$$

in which P is determined by the equation

$$P^2 = (k'\,cos\,\alpha + sn\,u\,dn\,u)^2 + k^2\,cn^4\,u.$$

We introduce for α an imaginary argument v.

We substitute

$$tg\,\tfrac{1}{2}\,\alpha = i\,\frac{tg\,\tfrac{1}{2}\,am\,v}{tg\,\tfrac{1}{2}\,am\,(u-K)}$$

and we find

$$sin\,\alpha = \frac{i\,sn\,v\,sn\,(u-K)}{cn\,v - cn\,(u-K)},$$

$$cos\,\alpha = \frac{1-cn\,v\,cn\,(u-K)}{cn\,v - cn\,(u-K)},$$

$$\frac{d\alpha}{sin\,\alpha} = \frac{dn\,v}{sn\,v}\,dv - \frac{dn\,(u-K)}{sn\,(u-K)}\,du,$$

$$P = \frac{cn^2\,u\,dn\,v\,dn\,(u-K)}{k'\,(cn\,v - cn\,(u-K))},$$

and finally

$$\frac{ds^2}{b^2} = \frac{dn^2\,v\,dn^2\,(u-K)}{k'^2(cnv - cn\,(u-K))^2}\,(du-dv)\,(du+dv).$$

From this ensues that $u+v$ and $u-v$ are the parameters of the lines of length zero, so that v is the parameter of the greatest incline.

According to the general properties of the minimal surfaces we have for the superficial element $d\Omega$ the expression

$$\frac{d\Omega}{b^2} = \frac{dn^2\, v\, dn^2\,(u-K)}{k'^2\,(cn\, v - cn\,(u-K))^2}\, du\, \frac{dv}{i},$$

and we find for that part of the surface limited by the two circles with the arguments $+u$ and $-u$:

$$\frac{\Omega}{4b^2} = \int_0^u du \int_0^{2iK'} \frac{dv}{i}\, \frac{dn^2\, v\, dn^2\,(u-K)}{k'^2\,(cn\, v - cn\,(u-K))^2}.$$

To perform the integration we start from the identity

$$f(u) = -\int_0^{2iK'} \frac{dv}{i}\, \frac{sn\,(u-K)\, dn\,(u-K)}{cn\, v - cn\,(u-K)} = 2K'\, Z(u-K) + \frac{\pi u}{K} =$$

$$= 2u\,(E'-K) + 2k'^2\, K'\, B\,(u),$$

which furnishes first

$$\int_0^{2iK'} \frac{dv}{i}\, \frac{dn^2\,(u-K)}{cn\, v - cn\,(u-K)} = \frac{k'\, f(u)}{cn\, u}$$

Moreover

$$'\!\!\int^{2iK'} \frac{dv}{i}\, \frac{dn^2\, v - dn^2\,(u-K)}{cn\, v - cn\,(u-K)} = k^2\; '\!\!\int_0^{2iK'} \frac{dv}{i}\, cn\, v + 2\, k^2\, K'\, cn\,(u-K).$$

A dash before the integral sign indicates that the path of integration does not pass through point $v = iK'$.

Out of the two last equations follows by means of addition

$$'\!\!\int_0^{2iK'} \frac{dv}{i} \cdot \frac{dn^2\, v}{cn\, v - cn\,(u-K)} = \frac{k' f(u)}{cn\, u} + k^2\; '\!\!\int_0^{2iK'} \frac{dv}{i}\, cn\, v + 2k^2 K'\, cn\,(u-K),$$

an equation which, if we differentiate with regard to u and then divide by $k'\, cn\, u$, passes into

$$\int_0^{2iK'} \frac{dv}{i} \cdot \frac{dn^2\, v\, dn^2\,(u-K)}{k'^2\,(cn\, v - cn\,(u-K))^2} = \frac{1}{cn\, u}\, \frac{d}{du}\left(\frac{f(u)}{cn\, u}\right) + \frac{2k^2 K'}{dn^2\, u}$$

$$= \frac{1}{2}\, \frac{d}{du}\left(\frac{f(u)}{cn^2\, u}\right) + \frac{2k^2 K'}{dn^2\, u} + \frac{k'^2 K'}{cn^2\, u\, dn^2\, u} + \frac{E'-K'}{cn^2\, u}.$$

Now integrating according to u between the limits 0 and u we find finally

$$\frac{\Omega}{4b^2} = \frac{u}{cn^2\, u}\,(E'-K') + E' A(u) + K'\, \frac{dn^2\, u}{cn^2\, u}\, B(u).$$

If the given circles have the radius $R = 1$ then b is equal to $cn\, u$ and we can write

$$\frac{\Omega}{4} = uE' + \xi\, cn\, u \frac{E' - K'}{k'} - sn^2\, uK'Q\, u),$$

where ξ again represents the x-coordinate of the centre M of the upper circle.

If this centre M moves on the envelope BA of the diagram, then u becomes equal to the critical argument u_0, $Q(u)$ equal to zero and we have obtained the greatest possible minimal surface Ω_0 for the given value of k. So

$$\frac{\Omega_0}{4} = u_0 E' + \xi_0\, cn\, u_0 \frac{E' - K'}{k'}.$$

We can now put the question where we have to put M on the envelope BA, that is what value must be given to k for Ω_0, to obtain the greatest possible value. To answer that question we substitute $c = k^2$ and $\varphi_0 = am\, u_0$, then Ω_0 is a function of c, whilst φ_0 and ξ_0 are connected with c by means of the equations

$$Q(u_0) = K - E - \int_{u_0}^{K} \frac{dw}{sn^2\, w} = 0,$$

$$\xi_0 = \sqrt{c}\, cnu_0\, A\,(u_0).$$

By differentiation we find

$$\frac{d\varphi_0}{dc} = -\frac{1}{2} sn^2\, u_0\, dn\, u_0\, B\,(u_0),$$

$$\frac{du_0}{dc} = -\frac{1}{2c} cn^2\, u_0\, A\,(u_0),$$

$$\frac{d\xi_0}{dc} = -\frac{1}{2\sqrt{c}} dn\, u_0\, B\,(u_0)\,(cn\, u_0\, dn\, u_0 + u_0\, c\, sn^2\, u_0),$$

and finally by means of these results

$$\frac{d}{dc}\left(\frac{\Omega_0}{4}\right) = \frac{K'-E'}{c'} cn\, u_0\, dn\, u_0\, B\,(u_0)\,(cn\, u_0\, dn\, u_0 + u_0\, c\, sn^3\, u_0).$$

As the right member of the last equation is always positive, Ω_0 always increases with c or with k. The greatest possible surface between the two circles is obtained by placing M in B; we have then a part of the catenoid, of which half the height is equal to $cot\,\beta = 0.6627$.

Now $\quad K' = E' = \frac{\pi}{2},$

$$\frac{\Omega_0}{2\pi} = u_0 = 1.1997.$$

The smallest value Ω_0 obtains for $k=0$. Then $\zeta_0=0$, $\xi_0=1$; the minimal surface consists only of the surface of the circles M and C' placed side by side in the XY-plane. We have

$$\frac{\Omega_0}{2\pi}=1.$$

So also the surface Ω_0 keeps moving between rather narrow limits. Although the value of Ω_0 depends again in rather an intricate way on k we can put pretty accurately, if once the critical argument u_0 or the amplitude φ_0 has been calculated,

$$\frac{\Omega_0}{2\pi}=\frac{1}{sn\, u_0}.$$

This is evident from the following table, in which have been inserted for some values of k the corresponding values of $\frac{\Omega_0}{2\pi}$ and of $\frac{1}{sn\, u_0}$.

k	$\frac{\Omega_0}{2\pi}$	$\frac{1}{sn\, u_0}$
Sin 0°	1.	1
15°	1 0002	1.0001
30°	1.0111	1 0176
45°	1 0556	1.0639
60°	1.1241	1 1271
75°	1.1795	1 1795
90°	1 1997	1 1997

As we have $b=cn\, u_0$, where b represents again the radius of the mean section we can in any case put with great approximation

$$\Omega_0=\frac{2\pi}{\sqrt{1-b^2}},$$

and in this way we obtain for the greatest possible just stable part of an arbitrary cyclic minimal surface that can be extended between two circles with radius $R=1$ the same expression as for the catenoid.