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are formed, on account of the early degeneration of the nucleus, which, by its divisions should have given rise to these nuclei. Further more, after the egg-apparatus has been formed, the remaining portion of the embryo-sac is only very slightly developed, so that there is no question of the formation of endosperm (what happens to the second generative nucleus, if indeed present, I have not been able to make out). It is much clearer here than in most cases, that this portion of the embryo-sac and the egg-cell are sister-cells. This agrees with the view of PORSCH [1]), according to whom the egg-apparatus of the higher plants is a reduced archegonium, the synergids being the neck canal-cells and the upper part of the embryo-sac with the upper polar nucleus being the ventral canal-cell. The latter hypothesis is however specially difficult in this case, for here the positions of egg-cell and of ventral canal-cell would be exactly reversed. A reduction in the antipodal apparatus, similar to that which occurs here, is found in *Helosis guyanensis*, according to the investigations of CHODAT and BERNARD [2]), and a still further reduction exists in *Cypripedium*, where, according to the researches of Miss PACE [3]), the lower portion of the embryo-sac has not even been laid down at all. It need scarcely be argued, that we are here concerned with a progressive differentiation, and not with the recurrence of ancestral peculiarities. Perhaps it may not be amiss to point out, in conclusion that we cannot here fall back for "explanation" on a parasitic or saprophytic mode of life of *Podostemaceae*.

Mathematics. — *"On twisted curves of genus two"*. By Prof. J. DE VRIES.

1. A curve of genus *two* bears one and only one involution of pairs of points I^2. On the nodal biquadratic plane curve it is determined by a pencil of rays, having the node as vertex; its coincidences are then the points of contact of the six rays touching the curve. If we could arrange the points of the curve in a second I^2 then this I^2 would be projected out of the node by a system of rays with correspondence [2], and the above six tangents would furnish six rays of ramification whilst a [2] can have four only.

[1]) O. PORSCH. Versuch einer phylogenetischen Erklärung des Embryosackes und der doppelten Befruchtung der Angiospermen. Jena 1907.

[2]) R. CHODAT et C. BERNARD. Sur le sac embryonnaire de l'Helosis guyanensis. Journal de Botanique T. XIV. 1900. p, 72.

[3]) LULA PACE. Fertilization in Cypripedium. Botanical Gazette. XLIV. 1907. p. 353.

2 We shall now consider the fundamental involution of pairs of points, F^2, on a twisted curve ϱ^n of genus two. It can be generated by a pencil of cones of order $(n-3)$. For, through an arbitrary point P pass $\frac{1}{2}(n-1)(n-2)-2$ bisecants of ϱ^n, and the cones of order $(n-3)$ through these $\frac{1}{2}(n-3)n-1$ right lines intersect ϱ^n in two variable points more.

This F^2 arranges the planes through the arbitrary right line a in an $[n]$-correspondence. If a is cut by a bisecant b bearing a pair of F^2, then the plane ab is a double coincidence of $[n]$, for it corresponds to $(n-2)$ planes with which it does not coincide. On the other hand each coincidence of F^2 determines a single coincidence of $[n]$. The number of double coincidences amounts thus to $\frac{1}{2}(2n-6)=n-3$, so that a is cut by $(n-3)$ bisecants b. In other words:

(*The right lines bearing the pairs of the fundamental involution form a scroll of order* $(n-3)$).

To determine the genus of this scroll φ^{n-3} we make use of a wellknown formula of ZEUTHEN. When there is between the points of two curves c and c' such a relation that to a point P of c correspond $\varkappa'$ points P' of c' and to a point P' correspond $\varkappa$ points P, whilst it happens y' times that two points P' and y times that two points P coincide, then the genus p and the genus p' of the curves are connected with the numbers mentioned before by the equation [1])

$$2\varkappa'(p-1)-2\varkappa(p'-1)=y-y'.$$

If now the points P and P^* of a pair of F^2 correspond to the point of intersection P' of the line connecting them and a fixed plane, then $p=2$, $\varkappa'=1$, $\varkappa=2$, $y'=0$, $y=6$, so $2-4(p'-1)=6$ and $p'=0$.

So the scroll φ^{n-3} is of *genus zero* and possesses therefore a *nodal curve* of order $\frac{1}{2}(n-4)(n-5)$.

For a ϱ^6 this involutory scroll is quadratic, so it is a *hyperboloid* or a *cone*.

In the former case one of the systems of generatrices consists of trisecants, the other of the bisecants bearing the pairs of F^2. The points of support of the trisecants are then arranged in the triplets of an involution which is likewise fundamental (i. o. w. given with the curve). That the latter has eight coincidences is easy to see from the (2,3)-correspondence between the two systems of generatrices.

By central projection we find a quadrinodal plane curve c^6, on

[1]) See ZEUTHEN, Math. Ann. III, 150. A simple proof has been given by KLUYVER (N. Archief v. W, XVII, 16).

which F^2 is cut by the conics containing the four nodes, whilst the lines connecting the pairs envelop a conic and at the same time bear the groups of a fundamental I^3 [1]).

If the involutory scroll of F^2 is a quadratic cone then every two pairs of F^2 lie in a plane through the vertex, which is at the same time a point of ϱ^5. This special ϱ^5 is evidently the section of a cubic surface and a quadratic cone, having a right line in common [2]).

4. We shall now consider a ϱ^6 of genus two. The involutory scroll of F^2 is now of order three (Φ^3). Let q be the double line, e the single director of Φ^3. As ϱ^6 lies on Φ^3 and a plane through q contains but one right line of Φ^3 which line bears a pair of F^2, we find that q has four points in common with ϱ^6, so it is a *quadrisecant.* So the fundamental involution is described by the pencil of planes having the quadrisecant as axis. From this is at the same time evident that ϱ^6 cannot have a second quadrisecant.

Each plane through e bears two pairs of F^2, so e is a chord of ϱ^6, and the pairs of F^2 are connected in pairs to form the groups of a particular I^4.

The planes connecting e with the two torsal right lines of Φ^3 are evidently double tangential planes of ϱ^6. On e therefore rest besides the tangents in the 6 coincidences of F^2 still the 4 tangents situated in those double tangential planes and the tangents to be counted double in the points of support of the chord e. The developable *surface of tangents* of ϱ^6 is therefore of order 14. This is evident also from the fact that the quadrisecant besides by the tangents in its points of support is intersected only by the six tangents of the coincidences.

By central projection out of a point of e we find a special c^6 with eight nodes of which the pairs of F^2 lie two by two on rays through a node which is at the same time the point of intersection of two nodal tangents.

5. The scroll β of the bisecants resting on a trisecant t is, like ϱ^6, of genus *two*. For, if the points B_1, B_2, B_3 of ϱ^6 lie with t in one plane then we can make each point B_k to correspond to the chord

[1]) A number of properties of c^5 are to be found in my paper: "Ueber Curven fünfter Ordnung mit vier Doppelpunkten" (Sitz. Ber. Akad. Wien, 1895, CIV, 46—59). The curves ϱ^5 and c^5 are treated by H. E. Timerding "Ueber eine Raumcurve fünfter Ordnung" (Journal f. d. r. u. a. Math., 1901, CXXIII, 284—311).

[2]) The central projection of this ϱ^5 has been treated in my paper quoted before page 63. It is generated by stating a projective correspondence between the rays of a pencil and the pairs of an involution, formed of the conics of a pencil.

$B_l B_m$, by which a (1, 1)-correspondence is determined between the points of ϱ^6 and the points of a plane section of the scroll β.

As each point of t evidently bears 5 bisecants, whilst a plane through t contains 3, the scroll β is a scroll of order 8. A plane section must now show singularities equivalent to 19 nodes. Now the intersection of t is a 5-fold point whilst the 6 intersections of ϱ^6 furnish as many nodes; the missing three nodes are evidently substituted by a threefold point which is the intersection of a trisecant resting on t.

So on the *scroll τ of the trisecants* these are arranged in pairs of an involution.

Furthermore follows from this that the scroll τ is of order 12. For, if x is the order of τ, then one of the $(x-1)$ points which t has in common with the remainder section in a plane laid through t is to be regarded as intersection of t; the remaining $(x-2)$ are derived from multiple curves. Now t is cut outside ϱ^6 by one trisecant and in each of its points of support by three trisecants, so $x-2=10$ and $x=12$.

6. Out of a point C of ϱ^6 we find F^2 projected on the curve in the triplets of an involution C^3.

For, if P is a point of ϱ^6 then the right line CP cuts the scroll Φ^3 in an other point F, and the plane through C, F and the point conjugate to it in F^2 determines on ϱ^6 two points P' and P'' more, forming with P an involutory group.

The planes $\pi \equiv PP'P''$ envelop a quadratic cone, namely the tangential cone of Φ^3 having C as vertex. A right line l through C is thus cut by two triplets of chords PP' situated in the two planes π through l; but moreover by the two chords connecting C with the two connecting points C' and C''. *The involutory scroll of C^3 is therefore of order eight.*

As we conjugate P to the chord $P'P''$ this scroll is also of genus *two*. In a plane section the point of intersection with ϱ^6 are nodes. From this ensues that there must be (see § 5) a *nodal curve* of order *thirteen*.

The central projection of ϱ^6 out of C is a quadrinodal c^5 upon which each group of C^3 is collinear to a pair of F^2. If we regard c^5 as central projection of a ϱ^5 then C^3 originates from the I^3 on the trisecants; consequently C^3 has like the last mentioned I^3 eight coincidences.

7. If we bring a cubic surface ψ^3 through 19 points of ϱ^6, this

curve lies on ψ^3, so it is the partial section of ψ^3 with the involutory scroll Φ^3. As q is nodal line of Φ^3 and single right line of ψ^3, the two surfaces have another line r in common. This r cannot coincide with the single directrix e, for then each right line of Φ^3 would have four points in common with ψ^3, viz: its points of intersection with ϱ^6, q and e; the surface ψ^3 would then however coincide with Φ^3.

Inversely we can regard ϱ^6 as section of a cubic scroll Φ^3 with nodal line q and a cubic surface ψ^3 having with Φ^3 the right line q in common and a right line r resting on the former one. A plane π through q cuts Φ^3 in a right line, ψ^3 in a conic, so it contains besides q two points of the curve of intersection, from which is evident that q is a quadrisecant; its points of support are coincidences of the $(1,4)$-correspondences between the points of contact of π with the two surfaces, one of the five coincidences is the point of intersection of q and r. That the single directrix of Φ^3 is a chord of ϱ^6, is evident f om the fact that it cuts ψ^3 on r, thus two times on ϱ^6.

8. If Φ^3 is replaced by a scroll of Cayley so that q is single directrix and at the same time generating line, then the conic of ψ^3, lying in the torsal tangential plane of Φ^3 determines on q two points each of which replaces in each plane π through q two points of intersection with ϱ^6; so they are *nodes* of ϱ^6. On this special curve the groups of F^2 are not arranged in pairs; for e coincides with q.

We obtain an other special ϱ^6 by taking instead of Φ^3 a cone with nodal edge q. The conics of ψ^3 situated in the planes touching Φ^3 along the nodal edge cut q in the points of support of the quadrisecant. Each edge of Φ^3 bears a pair of F^2, so that a plane through the vertex T contains three pairs.

The tangential cone out of T to ψ^3 has q and r as nodal edges; the six single edges which it has in common with Φ^3 are evidently tangents of ϱ^6 and contain the coincidences of F^2.

Through an arbitrary point O pass four tangential planes to Φ^3; the central projection of ϱ^6 furnishes a plane curve c^6 with four nodal tangents meeting in a single point C. The six single tangents out of C contain the coincidences of the fundamental involution, each ray of which through C bears three pairs. These are separated if we describe on F^2 a pencil of cubic curves having the eight nodes of c^6 as base-points.