Mathematics. — *"Some formulae concerning the integers less than n and prime to n."* By Prof. J. C. KLUYVER.

The number $\varphi(n)$ of the integers ν less than n and prime to n can be expressed by means of the divisors d.

We have

$$\varphi(n) = \sum_{d/n} \mu(d)\, d', \qquad (dd' = n)$$

if we denote by $\mu(q)$ the arithmetical function, which equals 0 if q be divisible by a square, and otherwise equals $+1$ or -1, according to q being a product of an even or of an odd number of prime numbers.

This equation is a particular case of a more general one, by means of which certain symmetrical functions of the integers ν are expressible as a function of the divisors d.

This general relation may be written as follows [1])

$$\sum_{\nu} f(\nu) = \sum_{d/n} \mu(d) \sum_{k=1}^{k=d'} f(kd).$$

For the proof we have to observe that, supposing $(m, n) \sim D$, the term $f(m)$ occurs at the righthand side as often as d in a divisor of D. Hence the total coefficient of the term $f(m)$ becomes

$$\sum_{d/D} \mu(d),$$

that is zero if D be greater than unity, and 1 when m is equal to one of the integers ν.

We will consider some simple cases of KRONECKER's equation.

First, let

$$f(y) = e^{xy}.$$

The equation becomes

$$\sum_{\nu} e^{x\nu} = \sum_{d/n} \mu(d) \sum_{k=1}^{k=d'} e^{xkd} = \sum_{d/n} \mu(d)\, e^{xd} \frac{e^{xn} - 1}{e^{xd} - 1},$$

or because of

$$\sum_{d/n} \mu(d) = 0,$$

$$\sum_{\nu} e^{x\nu} = \sum_{d/n} \mu(d) \frac{e^{xn} - 1}{e^{xd} - d}.$$

If we write

$$\sum_{\nu} \frac{x e^{x\nu}}{e^{xn} - 1} = \sum_{d/n} \mu(d) \frac{x}{e^{xd} - 1},$$

[1]) KRONECKER, Vorlesungen über Zahlentheorie. I, p. 251.

we may introduce the Bernoullian functions $f_k(\theta)$, defined by the equation

$$\frac{e^{\theta x}-1}{e^{x}-1} = \theta + \sum_{k=1}^{k=\infty} x^k f_k(\theta),$$

and hence show that

$$\sum_{\nu}\left\{\frac{1}{n} + \sum_{k=1}^{k=\infty} x^k n^{k-1} f'_k\left(\frac{\nu}{n}\right)\right\} = \sum_{d/n}\frac{\mu(d)}{d}\left\{1-\tfrac{1}{2}xd + \frac{B_1}{2!}x^2d^2 - \frac{B_2}{4!}x^4d^4 + \dots\right\}.$$

By equating the corresponding terms on the two sides we get

$$\sum_{\nu} f'_{2m}\left(\frac{\nu}{n}\right) = (-1)^{m-1}\frac{B_m}{2m!}\sum_{d/n}\mu(d)d'^{-2m+1}$$

as a first generalisation of the relation

$$\sum_{\nu}\nu^0 = \sum_{d/n}\mu(d)d'.$$

Observing that we have

$$\sum_{d/n}\mu(d)d'^{-2m+1} = \frac{1}{n^{2m-1}}\sum_{d/n}\mu(d)d^{2m-1},$$

there follows for two integers n and n', both having the same set of prime factors,

$$\frac{\sum_{\nu} f'_{2m}\left(\frac{\nu}{n}\right)}{\sum_{\nu'} f'_{2m}\left(\frac{\nu'}{n'}\right)} = \left(\frac{n'}{n}\right)^{2m-1}.$$

In the same way an expression for the sum of the k^{th} powers of the integers ν may be obtained. Expanding both sides of the equation

$$\sum_{\nu} e^{x\nu} = \sum_{d/n}\mu(d)\frac{e^{xn}-1}{e^{xd}-1}$$

we find

$$\frac{1}{k!}\sum_{\nu}\nu^k = \sum_{d/n}\mu(d)d^k f_k(d').$$

Other relations of the same kind, containing trigonometrical functions are deduced by changing x into $2\pi ix$.

From

$$\sum_{\nu} e^{2\pi ix\nu} = \sum_{d/n}\mu(d)\frac{e^{2\pi ixn}-1}{e^{2\pi ixd}-1}$$

we find by separating the real and imaginary parts

$$\sum_{\nu}\cos 2\pi x\nu = \tfrac{1}{2}\sin 2\pi xn\sum_{d/n}\mu(d)\cot \pi xd,$$

$$\sum_{\nu}\sin 2\pi x\nu = \sin^2 \pi xn\sum_{d/n}\mu(d)\cot \pi xd.$$

In particular the first of these equations gives a simple result if we put $x = \frac{1}{n} + \varepsilon$, where ε is a vanishing quantity. As the factor $\sin 2\pi x n$ tends to zero with ε the whole right-hand side is annulled but for the term in which $d = n$.

So it follows that

$$\sum_{\nu} \cos \frac{2\pi\nu}{n} = \mu(n),$$

and we have $\mu(n)$, originally depending upon the prime factors of n, expressed as a function of the integers prime to n.

Similarly we may put in the second equation $x = \frac{1}{2n}$ and write

$$\sum_{\nu} \frac{\sin \pi\nu}{n} = \sum_{d/n} \mu(d) \cot \frac{\pi d}{2n}.$$

Still another trigonometrical formula may be obtained by the substitution $x = \frac{q}{n} + \varepsilon$. Let D be the greatest common divisor of the integers n and q, so that

$$n = n_0 D \quad , \quad q = q_0 D;$$

then as ε vanishes, we have to retain at the right-hand side only those terms in which qd is divisible by n, or what is the same the terms for which the complementary divisor d' divides D.

Hence, we find

$$\sum_{\nu} \cos \frac{2\pi q\nu}{n} = \sum_{d'/D} \mu\left(\frac{n}{d'}\right) d' = D \sum_{d/D} \mu(n_0 d) \frac{1}{d}. \qquad (dd' = D)$$

Instead of extending the summation over all divisors d of D, it suffices to take into account only those divisors δ of n, that are prime to n_0. In this way we find

$$D \sum_{d/D} \mu(n_0 d) \frac{1}{d} = \mu(n_0)\, D \sum_{\delta} \mu(\delta) \frac{1}{\delta},$$

and as the second side is readily reduced to

$$\mu(n_0) \frac{\varphi(n)}{\varphi(n_0)} = \mu\left(\frac{u}{D}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{D}\right)},$$

we obtain for any integer q, for which we have $(n, q) \sim D$,

$$\sum_{\nu} \cos \frac{2\pi q\nu}{n} = \mu\left(\frac{u}{D}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{D}\right)}.$$

Concerning the result

$$\sum_{\nu}{}' \cos \frac{2\pi\nu}{n} = \mu\,(n)$$

a slight remark may be made. To each integer ν a second $\nu' = n - \nu$ is conjugated; hence denoting by ϱ_n an irreducible fraction $< \frac{1}{2}$ with the denominator n, we may write

$$2 \sum \cos 2\pi\varrho_n = \mu(n),$$

and also

$$2 \sum_{n \leq g} \cos 2\pi\varrho_n = \sum_{n \leq g} \mu(n).$$

Now for large values of g the fractions ϱ_n will spread themselves not homogeneously, but still with some regularity more or less all over the interval $0 - \frac{1}{2}$ and there is some reason to expect, that in the main the positive and the negative terms of the sum $\sum\limits_{n \leq g} \cos 2\pi\varrho_n$ will annul each other, hence the equation

$$2 \sum_{n \leq g} \cos 2\pi\varrho_n = \sum_{n \leq g} \mu(n) \cdot$$

is quite consistent with the supposition of VON STERNECK, that as g takes larger and larger values the absolute value of $\sum\limits_{n \leq g} \mu(n)$ does not exceed $\sqrt{g}$.

Another set of formulae will be obtained by substituting in KRONECKER'S equation

$$f(y) = \log \left(e^{\frac{2\pi i x}{n}} - e^{\frac{2\pi i y}{n}} \right).$$

Thus we get

$$\sum_{\nu} \log \left(e^{\frac{2\pi i x}{n}} - e^{\frac{2\pi i \nu}{n}} \right) = \sum_{d/n} \mu\,(d) \sum_{k=1}^{k=d'} \log \left(e^{\frac{2\pi i x}{n}} - e^{\frac{2\pi i k d}{n}} \right),$$

or

$$\sum_{\nu} \log \left(e^{\frac{2\pi i x}{n}} - e^{\frac{2\pi i \nu}{n}} \right) = \sum_{d/n} \mu(d) \log \left(e^{\frac{2\pi i x d'}{n}} - 1 \right)$$

and after some reductions

$$\sum_{\nu} \log 2 \sin \frac{\pi}{n} (\nu - x) = \sum_{d/n} \mu(d) \log 2 \sin \frac{\pi x}{d}.$$

By repeated differentiations with respect to x we may derive from this equation further analogies to the formula

$$\varphi\,(n) = \sum_{d/n} \mu\,(d)\, d'.$$

So for instance we obtain bij differentiating two times

$$\sum_{\nu} \frac{1}{\sin^2 \frac{\pi \nu}{n}} = \frac{1}{3} \sum_{d/n} \mu(d)\, d'^2$$

and by repeating the process

$$-\sum_{\nu} \left[\frac{d^{2m}}{dy^{2m}} \log \sin y\right]_{y=\frac{\pi\nu}{n}} = \frac{B_m\, 2^{2m}}{2m} \sum_{d/n} \mu(d)\, d'^{2m},$$

a result included in the still somewhat more general relation

$$n^s \sum_{\nu} \sum_{k=1}^{k=\infty} \frac{1}{(nk-\nu)^s} = \zeta(s) \sum_{d/n} \mu(d)\, d'^s,$$

which is self evident from.

Returning to the equation

$$\sum_{\nu} \log 2 \sin y \frac{\pi}{n} (\nu - x) = \sum_{d/n} \mu(d) \log 2 \sin \frac{\pi x}{d},$$

we obtain as x tends to zero

$$\sum_{\nu} \log 2 \sin \frac{\pi \nu}{n} = -\sum_{d/n} \mu(d) \log d.$$

In order to evaluate the right-hand side, we observe that for $n = p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots$ we have

$$-\sum_{d/n} \mu(d) \log d = -\left[\frac{d}{dy}(1 - e^{y \log p_1})(1 - e^{y \log p_2}) \ldots\right]_{y=0}.$$

So it is seen that, putting

$$-\sum_{d/n} \mu(d) \log d = \gamma(n),$$

the function $\gamma(n)$ is equal to zero for all integers n having distinct prime factors, and that it takes the value $\log p$, when n is any power of the prime number p.

Hence we may write

$$\prod_{\nu} 2 \sin \frac{\pi \nu}{n} = e^{\gamma(n)},$$

a result in a different way deduced by KRONECKER [1]).

Again in the equation

$$\prod_{\nu} 2 \sin \frac{\pi}{n} (\nu - x) = \prod_{d/n} \left(2 \sin \frac{\pi x}{d}\right)^{\mu(d)}$$

we will make x tend to $-\frac{n}{2}$.

If n be odd, all divisors d and d' are odd also and we have at once

[1]) KRONECKER, Vorlesungen über Zahlentheorie. I, p. 296,

$$\prod_{\nu} 2\cos\frac{\pi\nu}{n} = \prod_{d/n} (-1)^{\frac{d'-1}{2}\mu(d)} = (-1)^{\frac{1}{2}\varphi(n)}.$$

If $n = 2m$ and m be odd, we shall have $\varphi(m) = \varphi(n)$. Half the numbers $\varkappa$ prime to m and less than m will be equal to some integer ν, the other half will be of the form $\nu - m$.

Hence we have

$$\prod_{\nu} 2\sin\frac{2\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \prod_{\nu} 2\sin\frac{2\pi\varkappa}{n} = (-1)^{\frac{1}{2}\varphi(n)} \prod_{\varkappa} 2\sin\frac{\pi\varkappa}{m},$$

and therefore

$$\prod_{\nu} 2\cos\frac{\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \frac{\prod 2\sin\frac{\pi\varkappa}{m}}{\prod_{\nu} 2\sin\frac{\pi\nu}{n}} = (-1)^{\frac{1}{2}\varphi(n)} e^{\gamma\left(\frac{n}{2}\right) - \gamma(n)}$$

Lastly, if $n = 2m$, and m be even, we shall have $\varphi(m) = \frac{1}{2}\varphi(n)$. Now each of the numbers $\varkappa$ prime to m and less than m at the same time will be equal to some integer ν and to one of the differences $\nu - m$. Reasoning as before we have in this case

$$\prod_{\nu} 2\sin\frac{2\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \prod \left(2\sin\frac{2\pi\varkappa}{n}\right)^2 = (-1)^{\frac{1}{2}\varphi(n)} \prod_{\varkappa} \left(2\sin\frac{\pi\varkappa}{m}\right)^2.$$

and therefore

$$\prod_{\nu} 2\cos\frac{\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} \frac{\prod_{\nu}\left(2\sin\frac{\pi\varkappa}{m}\right)}{\prod 2\sin\frac{\pi\nu}{n}} = (-1)^{\frac{1}{2}\varphi(n)} e^{2\gamma\left(\frac{n}{2}\right) - \gamma(n)}.$$

From the foregoing we may conclude as follows. If we put

$$\prod 2\cos\frac{\pi\nu}{n} = (-1)^{\frac{1}{2}\varphi(n)} e^{\lambda(n)},$$

the arithmetical function $\lambda(n)$ is different from zero only when n is double the power of any prime number p, in which case we have $\lambda(n) = \log p$.

Again we introduce here the irreducible fractions ϱ_n less than $\frac{1}{2}$ with the denominator n; then denoting by $M(q)$ the least common multiple of all the integers not surpassing q we may write

$$2 \sum_{n \leqq g} \log 2\sin\pi\varrho_n = \sum_{n \leqq g} \gamma(n) = \log M(g),$$

$$2 \sum_{n \leqq g} \log 2\cos\pi\varrho_n = \sum_{n \leqq g} \lambda(n) = \log M\left(\frac{g}{2}\right).$$

27*

If we consider the quotient $\log M(g) : \log g$ as an approximate (but always too small) value of the number $A(g)$ of prime numbers less than g, to KRONECKER's result

$$A(g) = \frac{2}{\log g} \sum_{n \leqq g} \log 2 \sin \pi \varrho_n$$

we may add

$$A\left(\frac{g}{2}\right) = \frac{2}{\log \frac{g}{2}} \sum_{n \leqq g} \log 2 \cos \pi \varrho_n \,.$$

Astronomy. — *"Researches on the orbit of the periodic comet Holmes and on the perturbations of its elliptic motion.* IV." By Dr. H. J. ZWIERS. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

At the meeting of the Academy on the 27 January of 1906, a communication was made of my preliminary researches on the perturbations of the comet Holmes, during the period of its invisibility from January 1900 till January 1906, and also of an ephemeris of its apparent places from the 1st of May till the 31st of December 1906. This time again this computation led to its rediscovery. Owing to its large distance from the earth and the resulting faintness of its light, there seemed to be only a small chance for its observation during the first months. This proved to be true, as not before the 30th of August of this year, the Leiden observatory received a telegram, that the comet was found by prof. MAX WOLF at the observatory Koenigstuhl near Heidelberg, on a photograph taken in the night of the 28th of August of a part of the heavens where according to the ephemeris it ought to be found. The roughly measured place

$$\alpha = 61^\circ\ 51' \qquad \delta = +\ 42^\circ\ 28'$$

for $13^h\,52^m1$ local time, appeared to be in sufficient agreement with the calculation.

Afterwards the place of the comet has been twice photographically determined: on the 25th of September and on the 10th of October, and each time prof. WOLF was so kind, to communicate immediately to me the places as they had been obtained, after carefully measuring the plates. Although WOLF declared in a note to the observed