If we consider the quotient $log\, M(g) : log\, g$ as an approximate (but always too small) value of the number $A(g)$ of prime numbers less than $g$, to KRONECKER's result

$$A(g) = \frac{2}{log\, g} \sum_{n \leqq g} log\, 2\, sin\, \pi \varrho_n$$

we may add

$$A\left(\frac{g}{2}\right) = \frac{2}{log \frac{g}{2}} \sum_{n \leqq g} log\, 2\, cos\, \pi \varrho_n \, .$$

**Astronomy.** — *"Researches on the orbit of the periodic comet Holmes and on the perturbations of its elliptic motion. IV."* By Dr. H. J. ZWIERS. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

At the meeting of the Academy on the 27 January of 1906, a communication was made of my preliminary researches on the perturbations of the comet Holmes, during the period of its invisibility from January 1900 till January 1906, and also of an ephemeris of its apparent places from the 1st of May till the 31st of December 1906. This time again this computation led to its rediscovery. Owing to its large distance from the earth and the resulting faintness of its light, there seemed to be only a small chance for its observation during the first months. This proved to be true, as not before the 30th of August of this year, the Leiden observatory received a telegram, that the comet was found by prof. MAX WOLF at the observatory Koenigstuhl near Heidelberg, on a photograph taken in the night of the 28th of August of a part of the heavens where according to the ephemeris it ought to be found. The roughly measured place

$$\alpha = 61^\circ\, 51' \qquad \delta = +\, 42^\circ\, 28'$$

for $13^h\, 52^m 1$ local time, appeared to be in sufficient agreement with the calculation.

Afterwards the place of the comet has been twice photographically determined: on the 25th of September and on the 10th of October, and each time prof. WOLF was so kind, to communicate immediately to me the places as they had been obtained, after carefully measuring the plates. Although WOLF declared in a note to the observed position of the 25th of September[1]) that the brightness had increased sufficiently, to make the comet visible in a powerful telescope, till now I did not hear, that any visual observation of the comet has been made. The three Heidelberg plates are therefore the only material that can be used for testing the elements and ephemeris given before.

I communicate here the results as I had the pleasure to receive them from prof. WOLF.

1. "Den Kometen Holmes habe ich auf der Platte von 28 August rechtwinklig an die 4 Sterne

A.G. Bonn 3456, 3462, 3472, 3493

angeschlossen, und die Messungen nach der Turner'schen Methode reduziert. Ich finde für 1906.0:

$$\alpha = 4^h\ 7^m\ 34^s84 \qquad \delta = +\ 42^\circ\ 30'\ 59''9$$

für die Aufnahmezeit: 1906 Aug. 28, $13^h\ 52^m1$ Kgst. Das äusserst schwache zentrale Kernchen wurde dabei eingestellt. Die Messung und Rechnung bezieht sich auf die mittleren Orte der 4 Sterne für 1906; sonst ist gar nichts angebracht."

(Note of the 5th of September 1906).

2. "Ich habe Ihren Kometen nochmals am 25 aufgenommen und finde ihn entschieden etwas heller. Den Ort nach Turner mit 3 Sternen (A.G. Bonn 3710, 3760, 3778) fand ich

1906 Sept. 25: $12^h\ 46^m0$ M.Z. Kgst.

$$\alpha_{1906.0} = 4^h\ 32^m\ 10^s02 \qquad \delta_{1906.0} = +\ 47^\circ\ 34'\ 54''6$$

Ich habe auch den letzten Ort (viz. of Aug. 28) mit nur 3 Sternen nochmals gerechnet (weil ein Stern sehr ungünstig war) und fand für 1906 August 28: $13^h\ 52^m1$ Kgst.:

$$\alpha_{1906.0} = 4^h\ 7^m\ 35^s00 \qquad \delta_{1906.0} = +\ 42^\circ\ 30'\ 58''3$$

Ich bin nicht sicher, ob diese Bestimmung aus 3 Sternen besser ist als die erst mitgeteilte."

(Note of the 29th of September 1906).

3. "Herr Dr. KOPFF hat gestern den Ort einer Aufnahme vom 10 Okt. 1906 des Kometen Holmes ausgemessen .....

1906 Okt. 10: $9^h\ 1^m0$ Kgst.

$$\alpha_{1906.0} = 4^h\ 34^m\ 48^s94 \qquad \delta_{1906.0} = +\ 49^\circ\ 54'\ 59''2$$

Sterne: A.G. Bonn 3759, 3768, 3777 ..... Der Komet war

[1]) *Astron. Nachr.*, N$^o$. 4123, S. 302.

diesmal schon recht schwach, wahrnehmbar schwächer als im September. Die Messung ist deshalb auch wohl etwas unsicherer."

(Note of the 13th of October 1906).

Concerning the observation on the 28th of August I preferred the position obtained from 3 reference stars.

For the reduction to the apparent place, I used as before in the ephemeris the constants of the Nautical Almanac, where the short period terms are omitted. Assuming for the parallax of the sun 8″80, I find for the Heidelberg Observatory the following constants:

$$\lambda = -0^h 34^m 54^s 8$$
$$tg\,\varphi' = 0.06404$$
$$A = 9.58267$$
$$D = 0.82425$$

which are used for the computation of the parallax of the comet. The following table gives an account of the reduced observations.

T A B L E I.

| N⁰. | Red. on app. pl. | | Parallax | | Apparent geoc. place. | |
|---|---|---|---|---|---|---|
| | $\Delta\alpha$ | $\Delta\delta$ | $\Delta\alpha$ | $\Delta\delta$ | $\alpha$ | $\delta$ |
| | s | ″ | s | ″ | h m s | ° ′ ″ |
| 1 | + 1.888 | —8 55 | — 0.191 | +1 24 | 4 7 36.697 | +42 30 50.99 |
| 2 | + 2.929 | —8 57 | — 0 217 | +0.92 | 4 32 12.732 | +47 34 46.95 |
| 3 | + 3.593 | —7 51 | — 0.298 | +2.35 | 4 34 52.235 | +49 54 54.04 |

I used for comparison with the ephemeris my original computations, which contained in $\alpha$ as well as in $\delta$ one decimal place more than the published values. The computed places and their comparison with the observed positions, are given in the following table.

T A B L E II.

| Local time | Aberration-time. | Comp. apparent place | | Observ.—Comp. | |
|---|---|---|---|---|---|
| | | $\alpha$ | $\delta$ | $\alpha$ | $\delta$ |
| | d | h m s | ° ′ ″ | s | ″ |
| Aug. 28.553602 | 0.013211 | 4 7 29.753 | +42 30 24.28 | +6.94 | +26.7 |
| Sept. 25.507699 | .012005 | 4 32 4.255 | +47 34 29.94 | +8.48 | +17.0 |
| Oct. 10.351449 | .011462 | 4 34 43.017 | +49 54 43.02 | +9.22 | +11.0 |

Together with the ephemeris I communicated a table containing the variations of the right ascension and the declination by a variation of the perihelion passage of + 4 or — 4 days. In comparing the above given values $O—C$ with the numbers of that table, it is evident that by a small negative variation of the perihelion passage, the agreement between observation and computation may be nearly attained, at least in $\alpha$. The deviations in $\delta$ cannot be used so well for that purpose, as the variations of $\delta$, resulting from a variation of $T$, are always much smaller than those of $\alpha$, and this is especially the case in the period during which these observations are made. Yet we may conclude from the table for $\Delta T = -4$ days that the positive errors in $\delta$ will not entirely disappear by a variation of $T$.

By means of a rough interpolation I derived from the 3 differences $O—C$ in right ascension the following corrections for the time of perihelion passage:

| | | |
|---|---|---|
| Observ. of Aug. 28 : | $\Delta T = -0.0900$ | day |
| „ „ Sept. 25 : | $-0.0916$ | „ |
| „ „ Oct. 10 : | $-0.0896$ | „ |

In the average $\Delta T = -0.0904$ day, which at the rate of a mean daily motion of 517″448 corresponds to an increase of the mean anomalies of 46″8.

As a first step to correct the adopted elements of the orbit, I therefore computed the 3 places, in the supposition of an increase of the mean anomalies: 1° by 40″, 2° by 50″. I interpolated the following sun's co-ordinates (with reference to the mean equinox of 1906.0) from the Naut. Almanac.

TABLE III.

| 1906 | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| Aug. 28.540391 | — 0.9134887 | + 0.3947635 | + 0.1712510 |
| Sept. 25.495694 | — 1.0018399 | — 0.0318699 | — 0.0138250 |
| Oct. 10.339987 | — 0.9565810 | — 0.2616405 | — 0.1135029 |

For the reduction to the apparent places I added to the mean $\alpha$ of the comet: $f + g \sin (G + \alpha) \operatorname{tg} \delta$, to its mean $\delta$: $g \cos (G + \alpha)$. The following table contains the computed apparent places in the two suppositions.

TABLE IV.

| N⁰. | $\Delta M = +40''$ | | $\Delta M = +50''$ | |
|---|---|---|---|---|
| | $\alpha$ | $\delta$ | $\alpha$ | $\delta$ |
| | h m s | ° ′ ″ | h m s | ° ′ ″ |
| 1 | 4 7 35.758 | + 42 30 34 72 | 4 7 37.266 | + 42 30 37.38 |
| 2 | 4 32 11 451 | + 47 34 31 46 | 4 32 13 248 | + 47 34 31.85 |
| 3 | 4 34 51.050 | + 49 54 42 20 | 4 34 53 060 | + 49 54 41.99 |

A sufficient control is obtained here by comparing the values for $\Delta M = 0''$ (ephemeris), $\Delta M = +40''$ and $\Delta M = +50''$.

In comparing with the observed apparent places we obtain the following differences $O - C$:

TABLE V.

| N⁰. | $\Delta M = +40''$ | | $\Delta M = +50''$ | |
|---|---|---|---|---|
| | $\Delta\alpha$ | $\Delta\delta$ | $\Delta\alpha$ | $\Delta\delta$ |
| | s | ″ | s | ″ |
| 1 | + 0.939 | + 16.27 | — 0.569 | + 13.61 |
| 2 | + 1.281 | + 15.49 | — 0.516 | + 15.10 |
| 3 | + 1.185 | + 11.84 | — 0.825 | + 12.05 |

By means of interpolation between the values of $\Delta\alpha$ we find as resulting value for $\Delta M + 46''412$, leaving the following errors:

| N⁰. | $\Delta\alpha$ | $\Delta\delta$ |
|---|---|---|
| | s | ″ |
| 1 | — 0.03 | + 14.7 |
| 2 | + 0.13 | + 15.2 |
| 3 | — 0.10 | + 11.9 |

From this follows that by a variation of $M$ alone, the differences $O - C$ in $\alpha$ can be reduced to very small quantities, but this is not the case with the differences in $\delta$. It could be seen beforehand that no further improvement could be expected from alterations in $\pi$, $\varphi$ or $\mu$; at the end I will add a few words on these elements. So we must try to bring it about by variations in the position of the plane of the orbit, viz. of $i$ and ☊, and for this reason I determined the relation between those elements and the computed places of the comet. As from the two suppositions $\Delta M = +50''$ seems to be nearer to the truth, I computed the apparent places of the comet: for $\Delta M = +50''$ $\Delta i = +10''$ and $\Delta$☊ $= 0$ and also for $\Delta M = +50''$ $\Delta i = 0$ $\Delta$☊ $= -10''$. Probably a somewhat larger value of $\Delta$☊ had been more convenient. The following table gives the variations of $\alpha$ and $\delta$ in the two cases.

TABLE VI.

| N°. | $\Delta i = +10''$ | | $\Delta$☊ $= -10''$ | |
|---|---|---|---|---|
| | $\Delta\alpha$ | $\Delta\delta$ | $\Delta\alpha$ | $\Delta\delta$ |
| | s | '' | s | '' |
| 1 | $-0.149$ | $+10.00$ | $+0.040$ | $+1.26$ |
| 2 | $-0.108$ | $+11.95$ | $+0.067$ | $+0.83$ |
| 3 | $-0.111$ | $+12.88$ | $+0.080$ | $+0.56$ |

The numbers from the tables V and VI give the following values of the differential quotients of $\alpha$ and $\delta$ with respect to $M$, $i$ and ☊, which will be used as coefficients in the equations of condition.

| | Aug. 28 | Sept. 25 | Oct. 10 |
|---|---|---|---|
| $\frac{\partial\alpha}{\partial M}$ | $+0.1508$ | $+0.1797$ | $+0.2010$ |
| $\frac{\partial\delta}{\partial M}$ | $+0.266$ | $+0.039$ | $-0.021$ |
| $\frac{\partial\alpha}{\partial i}$ | $-0.0149$ | $-0.0108$ | $-0.0111$ |
| $\frac{\partial\delta}{\partial i}$ | $+1.000$ | $+1.195$ | $+1.288$ |
| $\frac{\partial\alpha}{\partial ☊}$ | $-0.0040$ | $-0.0067$ | $-0.0080$ |
| $\frac{\partial\delta}{\partial ☊}$ | $-0.126$ | $-0.083$ | $-0\ 056$ |

For $\alpha$ the second of time and for the others quantities the second of arc have been adopted as unities. I multiplied the equations of condition for $\alpha$ by $15 \cos \delta$, and instead of $\Delta ☊$ I introduced $\frac{\Delta ☊}{10}$ as unknown quantity.

*Equations of condition.*

*a.* From the Right ascensions:

$$0.22202\, \Delta M + 9.21681_n\, \Delta i + 9.64568_n \frac{\Delta ☊}{10} = 0.79873_n$$
$$0.25966 \;\;" \;\; + 9.03853_n \;\;" \;\; + 9.83118_n \;\;" \;\; = 0.71776_n$$
$$0.28811 \;\;" \;\; + 9.03023_n \;\;" \;\; + 9.88800_n \;\;" \;\; = 0.90136_n$$

*b.* From the Declinations:

$$9.42488\, \Delta M + 0.00000\;\; \Delta i + 0.10037_n \frac{\Delta ☊}{10} = 1.13386$$
$$8.59106 \;\;" \;\; + 0.07737 \;\;" \;\; + 9.91908_n \;\;" \;\; = 1.17898$$
$$8.32222_n \;\;" \;\; + 0.10992 \;\;" \;\; + 9.74819_n \;\;" \;\; = 1.08099$$

The coefficients are written logarithmically; the second members are taken from column 4 and 5 of table V, and therefore to $\Delta M$, found from these equations, the correction $+ 50''$ has still to be applied.

From the above equations of condition we derive in the ordinary way the following normal equations:

$$+ 9.9278\;\; \Delta M - 0.39596\, \Delta i - 3.8260 \frac{\Delta ☊}{10} = - 31.495$$
$$- 0.39596 \;\;" \;\; + 4.1375 \;\;" \;\; - 2.7434 \;\;" \;\; = + 49.637$$
$$- 3.8260 \;\;" \;\; - 2.7434 \;\;" \;\; + 3.8423 \;\;" \;\; = - 23.951$$

These equations are much simpler if we introduce besides $\Delta M$, only one of the two unknown quantities. If we try e.g. to represent the observations only through variations of $M$ and $i$ we have not only $\Delta ☊ = 0$ but the third equation falls out entirely.

1. *Solution for* $\Delta ☊ = 0$.

The results are:

$$\Delta M = - \;\; 2''\,7042$$
$$\Delta i \;= + 11.74$$

and the remaining errors:

| | | |
|---|---|---|
| 1. | $\Delta \alpha = + 0^s014$ | $\Delta \delta = + 2''59$ |
| 2. | $= + 0.097$ | $+ 1.18$ |
| 3. | $= - 0.151$ | $- 3.13$ |

2. *Solution for* $\Delta i = 0$.

In this case we find:

$$\Delta M = - \quad 9''0461$$
$$\Delta \Omega = - 2'32''41$$

and for the remaining errors:

| | | |
|---|---|---|
| 1. | $\Delta\alpha = + 0^s185$ | $\Delta\delta = - 3''18$ |
| 2. | $+ 0.089$ | $+ 2.80$ |
| 3. | $- 0.226$ | $+ 3.32$ |

3. *Solution with* 3 *unknown quantities:*

The results are:

$$\Delta M = - \quad 5''3045$$
$$\Delta i = + \quad 7.32$$
$$\Delta \Omega = - 1' 2.90$$

and according to the equations of condition there remain the following differences Obs.—Comp.

| | | |
|---|---|---|
| 1. | $\Delta\alpha = + 0^s088$ | $\Delta\delta = - 0''23$ |
| 2. | $+ 0\ 095$ | $+ 1.34$ |
| 3 | $- 0.181$ | $- 1.01$ |

As we see the solution with $\Delta\Omega = 0$ and that with $\Delta i = 0$ satisfy the observations fairly well, the first one somewhat better, especially in right ascension. Stili we cannot deny that in the values Obs.—Comp. of $\delta$ in both solutions, there exists a systematic variation. On account of that I prefer for the present the solution with 3 unknown quantities, where such a systematic variation doesnot appear. I therefore take the following elements as the most probable for the return in 1906:

Epoch 1906 January 16.0 M.T. Greenw.

$$M_0 = 1266456''838$$
$$= 351°47'36''838$$
$$\mu = 517''447665$$
$$\log a = 0.5574268$$
$$T = 1906 \text{ March } 14.09401$$
$$\varphi = 24°20'25''55$$
$$e = 0.4121574$$
$$\left.\begin{aligned} i &= 20°49'\ 0''62 \\ \pi &= 346\ \ 2\ 31.63 \\ \Omega &= 331\ 44\ 37.85 \end{aligned}\right\} 1906.0$$

Yet it is evident that the accuracy of these elements is not equal to the accuracy of those I could derive for previous returns of the comet. In the first place the observations include only a period of 43 days, in which the heliocentric motion of the comet with its large perihelion distance was not even 12°. Secondly three observations with their inevitable errors are in general only sufficient to obtain a mere approximate idea of the orbit. We must admire the ability and accuracy of the Heidelberg astronomers, who, from measurements on a short focal photographic plate taken of a still wholly invisible nebula, could deduce the position of the comet with an accuracy that could be compared to that of micrometer measurements of objects several hundred times brighter. Still we must bear in mind that the rejection of only one of the 4 reference stars on the plate of the 28th of August, had an influence of $0^s16$ in $\alpha$ and $1''6$ in declination, or of $2''39$ in arc of a great circle.

As a test to my calculations, I derived the 3 places finally by direct computation from the obtained elements.

*Heliocentric aequatorial co-ordinates:*

$$x = [9.993\ 7648.63]\, \sin (v + 77°37'28''36)$$
$$y = [9.876\ 2140.59]\, \sin (v - 20\ 58\ 46.82)$$
$$z = [9.832\ 7020.56]\, \sin (v - \ \ 1\ 46\ 46.76)$$

The following table contains the computed apparent places of the comet and the differences Obs.—Comp.

TABLE VII.

| N⁰. | α | δ | Δα | Δδ |
|---|---|---|---|---|
| | h m s | ° ′ ″ | s | ″ |
| 1 | 4 7 36.602 | + 42 30 51.32 | + 0.095 | — 0.33 |
| 2 | 4 32 12.633 | + 47 34 45.69 | + 0.099 | + 1.26 |
| 3 | 4 34 52.412 | + 49 54 55.19 | — 0.177 | — 1.15 |

The agreement between these differences found directly, and the quantities obtained by substitution in the equations of condition forms a sufficient control on the whole computation.

*The elements $\mu$, $\pi$ and $\varphi$.*

The elements from which the ephemeris for 1906 has been derived are those given in "Système VII" p. 78 of my Deuxième Mémoire, reduced to 1906 by applying the perturbations, arising from the action of Jupiter. The mean error of the obtained value for $\mu$ is so small, that although not absolutely impossible, it is hardly probable that the correction obtained for the mean anomaly should have been caused totally or for the greater part by an error in $\mu$. Taking the obtained $\Delta M$ for the 25th of Sept. we get:

$$\Delta \mu = + \frac{44'' 6955}{2662.50} = + 0'' 016787$$

and thus the real error of $\mu$ should be 67 times the mean one. Adopting this correction of $\mu$, the mean anomalies for the 28th of August and the 10th of October would be only 0″ 469 smaller and 0″ 249 greater than the adopted ones.

It is more probable that the correction of $M$ arises from neglected perturbations of that element by Saturn. This perturbation is given by the formula

$$\Delta M = \int_{t_0}^{t} \frac{dM}{dt}\, dt + \int\int_{t_0}^{t} \frac{d\mu}{dt}\, dt^2.$$

Even if instead of the sum of the values each term was known separately it would be equally impossible to conclude from the value of the double integral, the final value of $\int \frac{d\mu}{dt}\, dt$, or the correction of $\mu$ for 1906. Observations during a much longer period can only decide in this case.

Something like this holds for $\pi$ and $\varphi$. During the short period of the observations, we may even substitute for a part of the correction $\Delta M$ corresponding variations of $\pi$ and $\varphi$. If we keep to the plane of the orbit, the apparent place, except for small variations in the radius-vector (of little influence near the opposition), depends wholly on the longitude in the orbit, or on

$$l = \pi + v.$$

So we can apply small variations to the elements without varying perceptibly the computed positions, if only

$$\Delta l = \Delta \pi + \Delta v = 0$$

or

$$\Delta \pi = - \Delta v.$$

This relation provides us with the means to throw a part of the correction found for $M$ on $\pi$ or on $\varphi$ or on both together. In the first case we have to satisfy the equation

$$\Delta \pi = - \frac{\partial v}{\partial M} \Delta M.$$

We can derive the values of $\frac{\partial v}{\partial M}$ directly from the comparison of the two former computations with $\Delta M = +40''$ and $\Delta M = +50''$. And so I find for the three dates of the observations:

$$\Delta M = -0.506\,\Delta\pi$$
$$-0.549\,\Delta\pi$$
$$-0.573\,\Delta\pi$$

If we keep $\pi$ constant and want to substitute a part of the correction of $M$ by a variation of $\varphi$, we must satisfy the relation

$$\Delta v = 0$$

or

$$\Delta M = \left(\frac{\partial M}{\partial \varphi}\right)_{v\ \mathrm{const.}} \Delta\varphi.$$

I derived the values of $\left(\frac{\partial M}{\partial \varphi}\right)_{v\ \mathrm{const.}}$ by computing from the three values of $v$, with a varied excentricity, the corresponding values of the mean anomaly. Hence I got for the three observations:

$$\Delta M = -1.040\,\Delta\varphi$$
$$-1.186\,\Delta\varphi$$
$$-1.260\,\Delta\varphi$$

Although the coefficients as well those of $\Delta\pi$ as of $\Delta\varphi$ show a small variation in the influence of the corrections of the elements on the three positions, practically this influence differs too little from that of a constant variation of $M$ to allow a determination of $\Delta M$, $\Delta\varphi$ and $\Delta\pi$ separately from the three observations.

*Leiden*, November 1906.

**Mathematics.** "*On the locus of the pairs of common points and the envelope of the common chords of the curves of three pencils.*" (1st part). By Dr. F. SCHUH. (Communicated by Prof. P. H. SCHOUTE).

1. *Given three pencils* $(C_r)$, $(C_s)$, $(C_t)$ *of plane curves of degree* $r$, $s$, $t$. *To find the locus* $L$ *of the pairs of points through which passes a curve of each of those pencils.*

Let $P$ and $P'$ be the points of such a pair. When determining the locus we shall notice but those points $P$ and $P'$ which are for each couple of pencils *movable* points of intersection (i. e. points not necessarily coinciding with the basepoints), a distinction to be made only when the pencils have common basepoints. The locus $L$ arrived