*The envelope of the lines connecting pairs of points, through which a curve of each of the pencils is possible, is of class*

$$3\,rst - 2\,(st+tr+rs) + (r+s+t) - \alpha(r-1) - \beta(s-1) - \gamma(t-1) + \delta =$$
$$= 3\,rst - 2\,(st+tr+rs) + (r+s+t) - \alpha'(r-1) - \beta'(s-1) - \gamma'(t-1) -$$
$$- \delta(r+s+t-4).$$

8. If the pencils have no common basepoints then the class of the envelope is $3\,rst - 2\,(st + tr + rs) + (r + s + t)$. By a common basepoint $A_{st}$ of the pencils $(C_s)$ and $(C_t)$ that class is lowered with $r - 1$. *This is because point $A_{st}$ has separated itself from the envelope $r-1$ times.* In fact, the curve $C_r$ passing through $A_{st}$ has separated itself from the locus of the points $P$ and $P'$. If we take $P$ arbitrarily on this $C_r$, the corresponding point $P'$ coincides with $A_{st}$. So an arbitrary line passing through $A_{st}$ is to be regarded $(r-1)$ times as a line connecting $P$ and $P'$, as any of the $r-1$ points of intersection with $C_r$ differing from $A_{st}$ may be chosen for $P$.

If the three pencils have a common basepoint $A_{rst}$ the total envelope of $PP'$ remains definite (in contrast to the total locus of $P$ and $P'$). It is true $P$ can be taken quite arbitrarily, but then $P'$ coincides with a point $A_{rst}$, so that the line $PP'$ passes through that point $A_{rst}$, and therefore is not quite arbitrary. As the class of the envelope proper is lowered by the point $A_{rst}$ with $r + s + t - 4$ it follows, *that $A_{rst}$ separates itself $(r + s + t - 4)$ times from the envelope*. As one of the points of the pair becomes entirely indefinite, that multiplicity is not easy to explain, as far as I can see.

**Physics.** — *"On a new empiric spectral formula."* By E. E. Mogendorff. (Communicated by Prof. P. Zeeman).

By the fundamental investigations of Kayser and Runge and those of Rydberg the existence of spectral series was proved. The formulae of these physicists, however, give in general too great deviations for the first lines of a series. I have tried to improve the formula given by Rydberg:

$$n = A - \frac{N_0}{(m+a)^2}.$$

Particularly noteworthy in Rydberg's formula is the universal constant $N_0$. From Balmer's formula, which is included as a special case in Rydberg's formula, follows for hydrogen for the observation corrected to vacuo $N_0 = 109675$.



Assuming for a moment that the $N_0$ was also variable for the different series, I have calculated the constants $A$, $a$ and $N_0$ from three of the best observed curves. For $N_0$ the following values were found:

| | | |
|---|---|---|
| Principal Series | Lithium | 109996 |
| ,, ,, | Natrium | 107178 |
| ,, ,, | Potassium | 105638 |
| ,, ,, | Rubidium | 104723 |
| ,, ,, | Caesium | 104665 |
| 1st associated series | Hydrogen | 109704 |
| ,, ,, | Helium | 109703 |
| ,, ,, | Natrium | 110262 |
| ,, ,, | Potassium | 109081 |
| ,, ,, | Silver | 107162 |
| ,, ,, | Magnesium | 108695 |
| ,, ,, | Zinc | 107489 |
| ,, ,, | Oxygen | 110660 |
| Second ,, | Natrium | 107819 |
| ,, ,, | Magnesium | 105247 |
| ,, ,, | Calcium | 103702 |
| ,, ,, | Zinc | 105399 |
| ,, ,, | Aluminium | 105721 |

These values have been calculated from wave frequencies not corrected to vacuo.

As appears from these values $N_0$ is not absolutely constant. As Kayser [1]) found in another way, we see, however, that relatively

[1]) Kayser, Handbuch II. p. 553.

$N_0$ changes little from element to element [1]). The supposition lies at hand, that a constant of nature will occur in the rational formula. For the first associated series of Aluminium calculation gives a considerable deviation. Calculating from the first terms of this series we find $N_0 = 207620$ calculating from the middle lines $N_0 = 138032$, and from the lines with smaller $\lambda$ $N_0 = 125048$.

The first asssociated series of aluminium behaves therefore quite abnormally.

In RYDBERG's formula another function than $(m+a)^{-2}$ must be used to get a better harmony, specially with the first terms of a series.

In my thesis for the doctorate, which will shortly appear, I have examined the formula:

$$n = A - \frac{109675}{\left(m + a + \frac{b}{m}\right)^2},$$

in which $n$ represents the wave frequency reduced to vacuo, $A$, $a$ and $b$ are constants which are to be determined, $m$ passes through the series of the positive integers, starting with $m = 1$. In most cases with this formula a good agreement is obtained, also with the first lines of a series. The associated series converge pretty well to the same limit, while also the law of RYDBERG—SCHUSTER is satisfied in those cases where besides associated series, also a principal series is observed.

A spectral formula has also been proposed by RITZ [2]).

In my thesis for the doctorate I have adduced some objections to the formula of RITZ, as it gives rise to highly improbable combinations of lines. Moreover for the metals of the 2nd column of MENDELEJEFF's system his views are not at all in harmony with observation.

In the following tables the observed wavelength in Å. E. is given under $\lambda_w$, the limit of error of observation under F, the deviation according to the formula proposed by me under A, the deviation according to the formula of KAYSER and RUNGE under A. K. R. The mark * on the right above a wavelength indicates that these lines were used as a basis for the calculation of the constants A, $a$ and $b$.

The constants are calculated from the wave frequencies reduced to vacuo [3]).

[1]) The $B$ in KAYSER and RUNGE's formula varies within considerably wider limits than the $N_0$ of RYDBERG's formula.

[2]) Ann. d. Phys. Bd. 12, 1903, p. 264. W. RITZ, Zur Theorie der Serienspectren.

[3]) Where it was possible, I have always taken these values from the "Index of Spectra" from MARSHALL WATTS.



*Lithium.*

Principal series: A = 43480,13 ; $a = +0{,}95182$ ; $b = +0{,}00722$
1st ass. series : A = 28581,8 ; $a = +1{,}998774$ ; $b = -0{,}000822$
2nd ,, : A = 28581,8 ; $a = +1{,}59872$ ; $b = -0{,}00321$
3rd ,, ; A = 28581,8 ; $a = +1{,}95085$ ; $b = +0{,}00404$

The associated series converge here evidently to one limit.

The difference of wave frequency between the limits of principal and associated series is 43480,13—21581,8 = 14898,33. The wave frequency of the 1st line of the principal series is 14902,7. So the formula satisfies the law of RYDBERG-SCHUSTER pretty well.

PRINCIPAL SERIES.

| $m$ | $\lambda_W$ | F | K | A. K. R. |
|---|---|---|---|---|
| 1 | 6708,2 * | 0,20 | 0 | + 108 |
| 2 | 3232,77* | 0,03 | 0 | 0 |
| 3 | 2741,39 | 0,03 | — 0,06 | 0 |
| 4 | 2562,60* | 0,03 | 0 | 0 |
| 5 | 2475,13 | 0,10 | — 0,22 | — 0,2 |
| 6 | 2425,55 | 0,10 | — 0,18 | — 0,01 |
| 7 | 2394,54 | 0,20 | — 0,13 | + 0,30 |
| 8 | 2373,9 L. D. | ? | + 0,02 | + 0,75 |
| 9 | 2359,4 L. D. | ? | + 0,17 | + 1,18 |

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_W$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 6103,77* | 0,03 | 0 | 0 |
| 2 | 4602,37* | 0,10 | 0 | 0 |
| 3 | 4132,44 | 0,20 | — 0,11 | 0 |
| 4 | 3915,20* | 0,20 | 0 | — 0,20 |
| 5 | 3794,9 | 5,00 | + 0,09 | — 0,35 |
| 6 | 3718.9 | 5,00 | — 1,94 | — 2,25 |
| 7 | 3670,6 | 5,00 | — 1,06 | — 1,41 |

SECOND ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 8127,0* S | 0,30 | 0 | — 65 |
| 2 | 4972,11 | 0,10 | — 0,13 | 0 |
| 3 | 4273,44* | 0,20 | 0 | 0 |
| 4 | 3985,94 | 0,20 | + 0,22 | 0 |
| 5 | 3838,30 | 3,00 | + 2,40 | — 0,2 |

THIRD ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 6240,3* S | 0,40 | 0 | — |
| 2 | 4636,3* S | 0,40 | 0 | — |
| 3 | 4148,2 S | 1,00 | + 1,6 | — |
| 4 | 3921,8 E H | ? | — 0,88 | |

The capitals after the wavelengths denote the observers: L. D. LIVEING and DEWAR; S. SAUNDERS and E. H. EXNER and HASCHEK. Where no further indication is given, the observation has been made by KAYSER and RUNGE.

*Natrium.*

Principal series (the lines of the doublets with greatest $\lambda$)

$$A = 41447{,}09; \quad a = 1{,}147615; \quad b = -0{,}031484$$

Principal series (lines of the doublets with smallest $\lambda$)

$$A = 41445{,}20; \quad a = 1{,}148883; \quad b = -0{,}031908.$$

For the calculation of the limit of the associated series RYDBERG-SCHUSTER's law has been used. With a view to the constant differences of wave frequencies of the doublets of the associated series, I have only carried out the calculation for the components with small wavelength.

For the 1st ass. series $A = 24491{,}1; \; a = 1{,}98259; \; b = +0{,}00639$
For the 2nd ass. series $A = 24491{,}1; \; a = 1{,}65160; \; b = -0{,}01056$



PRINCIPAL SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 5896,16* | — | 0 | + 78 |
| 1 | 5890,19* | — | 0 | + 86 |
| 2 | 3303,07* | 0,03 | 0 | 0 |
| 2 | 3302,47* | 0,03 | 0 | 0 |
| 3 | 2852,91 | 0,05 | — 0,14 | 0 |
| 3 | 2852,91 | 0,05 | — 0,06 | 0 |
| 4 | 2680,46* | 0,10 | 0 | 0 |
| 4 | 2680,46* | 0,10 | 0 | 0 |
| 5 | 2593,98 | 0,10 | + 0,03 | + 0,03 |
| 5 | 2593,98 | 0,10 | — 0,02 | + 0,09 |
| 6 | 2543,85 L. D. | 0,10 | — 0,06 | + 0,10 |
| 6 | 2543,85 L. D. | 0,10 | — 0,14 | + 0,24 |
| 7 | 2512,23 L. D. | 0,20 | + 0,03 | + 0,50 |
| 7 | 2512,23 L. D. | 0,20 | — 0,10 | + 0,60 |

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 8184,33* L. | 0,2 | 0 | 0 |
| 2 | 5682,90 | 0,15 | 0,01 | 0 |
| 3 | 4979,30* | 0,20 | 0 | 0 |
| 4 | 4665,20 | 0,50 | — 0,13 | + 0,52 |
| 5 | 4494,30 | 1,00 | — 0,28 | + 0,50 |
| 6 | 4390,70 L. D. | ? | + 0,28 | + 1,30 |
| 7 | 4325,70 L. D. | ? | + 4,00 | + 1,76 |

SECOND ASSOCIATED SERIES.

| | | | | |
|---|---|---|---|---|
| 1 | 11404 | ? | + 1,00 | + 100,– |
| 2 | 6154,62* | 0,10 | 0 | 0 |
| 3 | 5149,19* | 0,10 | 0 | 0 |
| 4 | 4748,36 | 0,15 | + 0,12 | 0 |
| 5 | 4542,75 | 0,20 | + 0,65 | + 1,39 |
| 6 | 4420,20 L. D. | ? | + 0,02 | + 1,55 |
| 7 | 4343,70 | ? | + 2,00 | — 1,36 |

*Zinc.*

For this element I have calculated the formulae of the 1st and 2nd associated series for the components with the greatest wavelength of the triplets.

The limits are determined for the two series separately, for the first associated series the calculation gave 42876,25 and for the second associated series the limit appeared to be 42876,70. A very good agreement.

The formula gives as 1st line of the 1st associated series of Zinc the line 8024,05, which has not been observed. The 8th line of the first associated series 2409,22 has not been observed either. As 9th line of this series 2393,93 was calculated, which is in remarkably good harmony with the intense line 2393,88. As yet this line had not yet been fitted in the series. The great intensity of a curve in the root of the series is certainly strange; an investigation of the magnetic splitting might decide whether it is correct to range this line under the first associated series.

The formula for the 1st associated series is:

$$n = 42876{,}25 - \frac{109675}{\left(m + 0{,}909103 - \frac{0{,}007085}{m}\right)^2}$$

and for the 2nd associated series:

$$n = 42876{,}70 - \frac{109675}{\left(m + 1{,}286822 - \frac{0{,}058916}{m}\right)^2}$$

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | —— | — | — | — |
| 2 | 3345,13* | 0,03 | 0 | — 0,08 |
| 3 | 2801,00* | 0,03 | 0 | + 0,03 |
| 4 | 2608,65* | 0,05 | 0 | + 0,06 |
| 5 | 2516,00 | 0,20 | + 0,04 | — 0,11 |
| 6 | 2463,47 | 0,20 | — 0,14 | — 0,39 |
| 7 | 2430,74 | 0,30 | + 0,22 | + 9,00 |
| 8 | —— | — | — | — |
| 9 | 2393,88 | 0,05 | — 0,05 | — |



SECOND ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 4810,71* | 0,03 | 0 | + 58 |
| 2 | 3072,19* | 0,05 | 0 | 0,00 |
| 3 | 2712,60* | 0,05 | 0 | + 0,02 |
| 4 | 2567,99* | 0,10 | + 0,11 | — 0,01 |
| | 2493,67 | 0,15 | + 0,12 | — 0,04 |
| 6 | 2449,76 | 0,25 | — 0,11 | — 0,20 |

*Thallium.*

The formula for the 1st associated series is:

$$n = 41466{,}4 - \frac{109675}{\left(m + 1{,}90141 - \frac{0{,}00366}{m}\right)}$$

for the satellites:

$$n = 41466{,}4 - \frac{109675}{\left(m + 1{,}88956 - \frac{0{,}00085}{m}\right)^2}$$

and for the second associated series:

$$n = 41466{,}4 - \frac{109675}{\left(m + 1{,}26516 - \frac{0{,}07108}{m}\right)^2}$$

The limit has been calculated from three lines of the 1st associated series; only two more lines were required of the satellites and of the 2nd associated series. So in this spectrum all the constantshavebeen calculated from 7 lines and 31 lines are very well represented by the formula.

FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 3519,39* | 0,03 | 0 | — |
| 2 | 2918,43 | 0,03 | — 0,04 | — |
| 3 | 2709,33* | 0,03 | 0 | — |
| 4 | 2609,08 | 0,03 | + 0,04 | — |
| 5 | 2552,62* | 0,10 | 0 | — |
| 6 | 2517,50 | 0,10 | — 0,06 | — 0,34 |
| 7 | 2494,00 | 0,10 | — 0,03 | — 0,19 |
| 8 | 2477,58 | 0,10 | — 0,09 | + 0,06 |
| 9 | 2465,54 | 0,20 | — 0,17 | + 0,24 |
| 10 | 2456,53 | 0,20 | — 0,15 | + 0,47 |
| 11 | 2449,57 | 0,30 | — 0,17 | + 0,68 |
| 12 | 2444,00 | 0,30 | — 0,28 | + 0,79 |
| 13 | 2439,58 | 0,30 | — 0,24 | + 0,95 |

SATELLITES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 3529,58* | 0,03 | 0 | + 0,02 |
| 2 | 2921,63 | 0,03 | + 0,06 | — 0,07 |
| 3 | 2710,77* | 0,03 | 0 | + 0,13 |
| 4 | 2609,86 | 0,03 | — 0,03 | — 0,02 |
| 5 | 2553,07 | 0,10 | — 0,05 | — 0,12 |

SECOND ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 5350,65* | 0,03 | 0 | — 168 |
| 2 | 3229,88* | 0,03 | 0 | — 21,7 |
| 3 | 2826,27 | 0,05 | — 0,05 | — 3,65 |
| 4 | 2665,67 | 0,05 | — 1,32 | — 1,69 |
| 5 | 2585,68 | 0,05 | — 0,16 | + 0,01 |
| 6 | 2538,27 | 0,10 | — 0,17 | + 0,04 |
| 7 | 2508,03 | 0,15 | — 0,14 | — 0,01 |
| 8 | 2487,57 | 0,20 | — 0,06 | + 0,08 |
| 9 | 2427,65 | 0,20 | — 0,34 | — 0,21 |
| 10 | 2462,01 | 0,30 | — 0,20 | — 0,03 |
| 11 | 2453,87 | 0,30 | — 0,17 | + 0,07 |
| 12 | 2447,59 | 0,30 | — 0,05 | + 0,22 |
| 13 | 2442,24 | 0,30 | — 0,37 | — 0,01 |

I shall just add a few words on the spectrum of Aluminium. None of the formulae given as yet represents the first associated series of this element at all satisfactorily; nor is a satisfactory result attained with my formula. In the beginning of this paper I have pointed out, that very deviating values for $N_0$ were calculated from three of the 1st lines of the series.

The formula runs:

$$n = 48287{,}9 - \frac{109675}{\left(m + 0{,}89436 + \frac{1{,}038060}{m}\right)^2}$$



The constants have been calculated from the lines 4, 5 and 6.

ALUMINIUM. FIRST ASSOCIATED SERIES.

| $m$ | $\lambda_w$ | F | A | A. K. R. |
|---|---|---|---|---|
| 1 | 3082,27 | 0,03 | — 268,82 | + 384,8 |
| 2 | 2568,08 | 0,03 | + 3,46 | + 53,5 |
| 3 | 2367,16 | 0,03 | + 2,52 | + 6,1 |
| 4 | 2263,83* | 0,10 | 0 | + 0,03 |
| 5 | 2204,73* | 0,10 | 0 | + 0,17 |
| 6 | 2168,87* | 0,10 | 0 | — 0,13 |
| 7 | 2145,48 | 0,20 | + 0,06 | — 0,31 |
| 8 | 2129,52 | 0,20 | + 0,11 | — 0,21 |
| 9 | 2118,58 | 0,20 | — 0,28 | + 0,44 |

The agreement with the first lines (1, 2 and 3), leaves much to be desired. The value of the constant $b$ is here 1,03806, greater than the value of $a$ in that formula; this does not occur with any of the other series.

With 4 constants, so with:

$$n = A - \frac{109675}{\left(m + a + \frac{b}{m} + \frac{c}{m^2}\right)^2}$$

a better result is most likely reached. When the constants $b$ and here probably also the $c$, are not small with respect to $a$, then the influence of those constants is very great, particularly for small values of $m$. The deviation for the first line of the above series (3082,27), however, is so great, that I doubt if this is really the first line of this series.

The behaviour of this Aluminium series is certainly peculiar, and a further investigation is desirable.

For the way in which the constants in the formula were calculated, and for the spectra of Potassium, Rubidium and Calcium, of Magnesium, Calcium, Cadmium and of Helium and Oxygen, I refer to my thesis for the doctorate, which will shortly be published.