So therefore

$$V_I + I_{II} = \tfrac{1}{4}\pi\,\alpha_{23} + \tfrac{1}{4}\pi\,\alpha_{12} - \tfrac{1}{2}\alpha_{12}\left(\tfrac{1}{2}\pi - \alpha_{34}\right) + \text{ constant.}$$

The constant is found by putting α_{12} equal to $\alpha_{23} = \alpha_{34} = \alpha_{12} = \frac{1}{2}\pi$, in which case V_I takes up the sixteenth part of the whole hypersphere, i. e. $\frac{1}{8}\pi^2$, whilst V_{II} becomes $= 0$.

The constant then proves to be $-\frac{1}{8}\pi^2$, hence

$$V_I + V_{II} = -\tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\,\alpha_{12} + \tfrac{1}{4}\pi\,\alpha_{23} - \tfrac{1}{2}\alpha_{12}\left(\tfrac{1}{2}\pi - \alpha_{34}\right).$$

Likewise we find.

$$V_{II} + V_{III} = -\tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\left(\tfrac{1}{2}\pi - \alpha_{12}\right) + \tfrac{1}{4}\pi\,\alpha_{12} - \tfrac{1}{2}\left(\tfrac{1}{2}\pi - \alpha_{14}\right)\left(\tfrac{1}{2}\pi - \alpha_{23}\right),$$
$$V_{III} + V_{IV} = -\tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\,\alpha_{14} + \tfrac{1}{4}\pi\left(\tfrac{1}{2}\pi - \alpha_{12}\right) - \tfrac{1}{2}\alpha_{34}\left(\tfrac{1}{2}\pi - \alpha_{12}\right), \text{ etc.}$$

Every time the sum of the volumes of two successive tetrahedra can be expressed by means of four successive elements of the first cycle mentioned in § 4. We deduce easily from this:

$$V_I - V_{III} = \tfrac{1}{2}\alpha_{12}\alpha_{34} - \tfrac{1}{2}\alpha_{14}\left(\tfrac{1}{2}\pi - \alpha_{23}\right),$$

whilst in like manner we can find $V_{II} - V_{IV}$, $V_{III} - V_V$, etc. Further we find

$$V_I + V_{IV} = \tfrac{1}{2}\alpha_{14}\alpha_{23} - \tfrac{1}{2}\alpha_{34}\left(\tfrac{1}{2}\pi - \alpha_{12}\right) - \tfrac{1}{2}\alpha_{12}\left(\tfrac{1}{2}\pi - \alpha_{34}\right)$$

and in like manner $V_{II} + V_V$, etc.

If we remember that the tetrahedra I and VII are alike with respect to their elements and volumes, II and VIII also, etc. and that with respect to the volumes we have to deal with only a closed range of six terms we see that of each arbitrary pair always either the sum or the difference of the volumes can be expressed in a simple manner.

Mathematics. — *"The locus of the cusps of a threefold infinite linear system of plane cubics with six basepoints."* By Prof. P. H. Schoute.

In the generally known representation of a cubic surface S^3 on a plane α to the plane sections of S^3 correspond the cubics through six points in α; here to the parabolic curve s^{12} of S^3 answers the locus C^{12} of the cusps of the linear system of those cubics. The principal aim of this short study is to deduce from wellknown properties of s^{12} properties of c^{12} and reversely.

1. If a plane rotates around a right line l of S^3 the points of intersection of that line l with the completing conic describe on l an involution, the double points of which are called the asymptotic points of l. According to the condition of reality of these asymptotic

points the 27 right lines of S^3, supposed to be real, are to be divided into two groups, into a group of 12 lines with imaginary asymptotic points, the lines

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{vmatrix}$$

of a doublesix and into a group of 15 lines $c_{12}, c_{13}, \ldots, c_{56}$ with real asymptotic points. If to the six basepoints A'_i of the linear system of the cubics the six lines a_i correspond — and this case we shall in the following continually have in view — then to the six lines b_i correspond the six conics $b_i'^2$ through all the basepoints except A'_i and to the fifteen lines c_{ik} correspond the connecting lines $c'_{ik} = (A'_i, A'_k)$, whilst to the systems of conics $(\overset{2}{a_i})$ in planes through a_i, $(\overset{2}{b_i})$ in planes through b_i, $(\overset{2}{c_{ik}})$ in planes through c_{ik} correspond successively the pencils of the curves of the linear system with A'_i as doublepoint, the lines (b'_i) through A'_i and the conics $(c_{ik}'^2)$ through the four basepoints differing from A_i and A_k. The situation of the six points A'_i is then such that each of the fifteen lines c'_{ik} is touched in real points by two conics of the pencils $(c_{ik}'^2)$, whilst on the other hand the points of contact of the tangents out of the points A'_i to the conics $b_i'^2$ are imaginary so that each point A'_i lies within the conic $b_i'^2$ with the same index.

2. As a matter of fact all real points of a line l of S^3 are hyperbolic points of this surface with the exception of the two asymptotic points of this line showing a parabolic character; whilst each of these asymptotic points is point of contact of l with a conic lying on S^3, l touches in both points the parabolic curve S^{12}. If we apply this to each of the six lines a_i, imaged in the points A'_i, and if we consider that to a definite point P of a_i corresponds the point P' lying infinitely close to A'_i connected with A'_i by a line of definite direction (Versl., vol. I, pag. 143) we find immediately:

"The six basepoints A'_i of the linear system are fourfold points of the curve c^{12} of a particular character, consisting of the combination of two real cusps with conjugate imaginary cuspidal tangents, the cuspidal tangents of the curves out of the system with a cusp in A'_i".

The twelve points of intersection of the line c'_{ik} with c^{12} consist of the isolated points A'_i, A'_k counting four times and the real points of contact with two conics out of the pencil (c'^2_{ik}) counting two times. Likewise do the 24 points of intersection of the conic b'^2_i with c^{12} consist of the five basepoints differing from A'_i counting four times and the imaginary points of contact with the tangents through A'_i counting two times.

3. From the investigations of F. KLEIN and H. G. ZEUTHEN dating from 1873 and 1875 it has become evident that the surface S^3 with 27 real right lines has ten openings and the parabolic curve s^{12} has ten oval branches. In connection with this we find:

"The locus c^{12} has ten oval branches."

We ask which situation of the six basepoints A'_i corresponds to the particular case of the "surface of diagonals" of CLEBSCH, in which the ten oval branches of the curve s^{12} have contracted to isolated points. In this case the fifteen lines with real asymptotic points, i.e. in our case the lines c'_{ik}, pass ten times three by three through a point; this is satisfied by the six points consisting of the five vertices of a regular pentagon and the centre of the circumscribed circle.

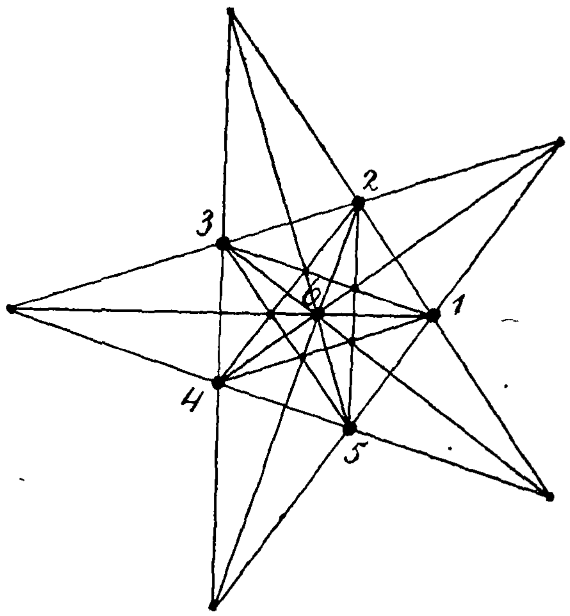

Fig. 1.

What is more, each six points having the indicated situation can be brought by central projection to this more regular shape. The ten meeting-points of the triplets of lines then form the vertices of two regular pentagons (fig. 1). The curve c^{12} corresponding to these six basepoints then consists of merely isolated points, namely of fourfold points in the six basepoints and twofold points in the ten meeting points of the triplets of lines.

The remark that the curve c^{12} belonging to the six basepoints of fig. 1 has the line c'_{16} as axis of symmetry and transforms itself into itself when rotating 72° around A'_6, enables us to deduce in a simple way its equation with respect to a rectangular system of coordinates with A'_6 as origin and c'_{16} as x-axis. The forms which pass into themselves by the indicated rotation are

$$\varrho^2 = x^2 + y^2,\ P = x^5 - 10x^3y^2 + 5xy^4,\ Q = 5x^4y - 10x^2y^3 + y^5.$$

If we pay attention to the axis of symmetry and to the identity $P^2 + Q^2 = \varrho^{10}$ the indicated equation can be written in the form

$$\varrho^4 + a\varrho^6 + b\varrho^8 + c\varrho^{10} + d\varrho^{12} + P(e + f\varrho^2 + g\varrho^4 + h\varrho^6) + P^2(i + k\varrho^2) = 0,$$

so that we have to determine only the ten coefficients $a, b, \ldots, k$. If now the common distance of the points $A'_1, A'_2, \ldots, A'_5$ to A'_6 is unity, then

$$x^4\left(x + \frac{3-e}{2}\right)^2 (x-1)^4 \left(x + \frac{3+e}{2}\right)^2 = 0,$$

where e stands for $\sqrt{5}$, represents the twelve points of intersection of the curve with the x-axis. By performing the multiplication this passes into

$$x^4(x^8 + 2x^7 - 7x^6 - 6x^5 + 20x^4 - 6x^3 - 7x^2 + 2x + 1) = 0.$$

From this follows

$$\begin{array}{llll} a = -7, & b = 20, & c + i = -7, & d + k = 1, \\ e = 2, & f = -6, & g = -6, & h = 2. \end{array}$$

So the equation

$$\varrho^4 - 7\varrho^6 + 20\varrho^8 - 7\varrho^{10} + \varrho^{12} + 2P(1 - 3\varrho^2 - 3\varrho^4 + \varrho^6) - Q^2(i + k\varrho^2) = 0$$

is determined, with the exception of the coefficients i and k still unknown. Now the parallel displacement of the system of coordinates to A'_1 as origin furnishes a new equation, of which the form

$$(4 - i - k)y^2 + 2(12 - 4i - 5k)xy^2 + x^4 + (54 - 28i - 45k)x^2y^2 + (5 + 4i + 3k)y^4$$

represents, after multiplication by 25, the terms of a lower order than five. The new origin being a fourfold point of c^{12} and the terms with y^2 and xy^2 having thus to vanish, we find

$$i = 8 \quad , \quad k = -4,$$

on account of which the indicated form passes into

$$(x^2 + 5y^2)^2.$$

The correctness of this result is evident from the following. Just as the two tangents in the old origin counting two times are represented by $x^2 + y^2 = 0$, and therefore coincide with the tangents out of A'_6 to the conic through the other basepoints, so $x^2 + 5y^2 = 0$ represents for the new origin A'_1 the pair of tangents out of A'_1 to the conic through the other basepoints. Or, if one likes, just as $x^2 + y^2$ is with the exception of a numerical factor, the fourth transformation ("Ueberschiebung") of the first member of the equation $Q = 0$ of the lines connecting A'_6 to the remaining basepoints, so $x^2 + 5y^2$ represents, likewise with the exception of a numerical factor, the fourth transformation of the first member of the equation $\frac{Py}{x} = 0$, which indicates with respect to the new origin A'_1 the five lines connecting A'_1 to the remaining basepoints.

Finally the equation of c^{12} is

$$\varrho^4(\varrho^8-7\varrho^6+20\varrho^4-7\varrho^2+1)+2P(\varrho^6-3\varrho^4-3\varrho^2+1)+4Q^2(\varrho^2-2)=0, \quad (1)$$

or entirely in polar coordinates (ϱ, φ)

$$4(\varrho^2-2)\varrho^5 \cos 5\varphi=(\varrho^2+1)(\varrho^4-4\varrho^2+1)\pm(\varrho^2-1)^2\sqrt{(\varrho^2-1)(4\varrho^2+1)(5\varrho^2-1)}. \quad (2)$$

It is easy to show that this curve admits of no real points differing from the six basepoints A'_i and the ten points of intersection of the triplets of connecting lines. If for brevity we write (2) in the form

$$L \cos 5\varphi = M \pm \sqrt{N},$$

then we find

$$- L^2 \sin^2 5\varphi = (M^2 + N - L^2) \pm 2M\sqrt{N} \quad . \quad . \quad . \quad (3)$$

and

$$\left.\begin{array}{l} M^2 + N - L^2 \equiv 2(\varrho^2-1)^2(2\varrho^2-1)(\varrho^8-6\varrho^6+14\varrho^4+2\varrho^2-1) \\ (M^2+N-L^2)^2 - 4M^2 N \equiv 4\varrho^8(\varrho^2-1)^4(\varrho^2-2)^2(\varrho^4-7\varrho^2+1)^2 \end{array}\right\} \quad . \quad . \quad (4)$$

If now we moreover notice that N is negative and therefore $\cos 5\varphi$ complex when ϱ^2 lies between $\frac{1}{5}$ and 1, the following is immediately evident:

a. The first member of the second equation (4) tends to zero, when ϱ^2 assumes one of the values $0, 1, 2, \frac{1}{2}(7 \pm 3e)$; it is positive for all other values of ϱ^2.

b. If $\sqrt{N}$ is real and ϱ^2 differs from unity the second member of the first equation (4) is positive; for the equation

$$\varrho^8 - 6\varrho^6 + 14\varrho^4 + 2\varrho^2 - 1 = 0$$

has, as is evident when the roots ϱ^2 are diminished by $1\frac{1}{2}$, besides *one* real negative root only *one* real positive one between $\frac{1}{5}$ and 1.

c. If ϱ^2 differs from $0, 1, 2, \frac{1}{2}(7 \pm 3e)$ the second member of

(3) is positive when N is positive, and therefore φ is imaginary.

d. Neither does $\rho^2 = 2$ give a real value for φ; for substitution in (1) furnishes for $\cos 5\varphi$ the result $\frac{3}{4}\sqrt{2}$.

e. So we find only the real points:

$\rho = 0$, φ indefinite . . . . . A'_6,

$\rho = 1$, $\cos 5\varphi = 1$ . . . . . . A'_1, A'_2, A'_3, A'_4, A'_5,

$\rho = \frac{3 \pm e}{2}$, $\cos 5\varphi = -1$ . . . the ten points of intersection of the fifteen connecting lines three by three.

4. We now consider a second case, in which the position of the six basepoints is likewise a very particular one, where namely these points form the vertices of a complete quadilateral. Through these six points not one genuine cubic with a cusp passes. For the three pairs of opposite vertices (A_1, A_2), (B_1, B_2), (C_1, C_2) of a complete quadilateral (fig. 2) form on each curve of order three, containing

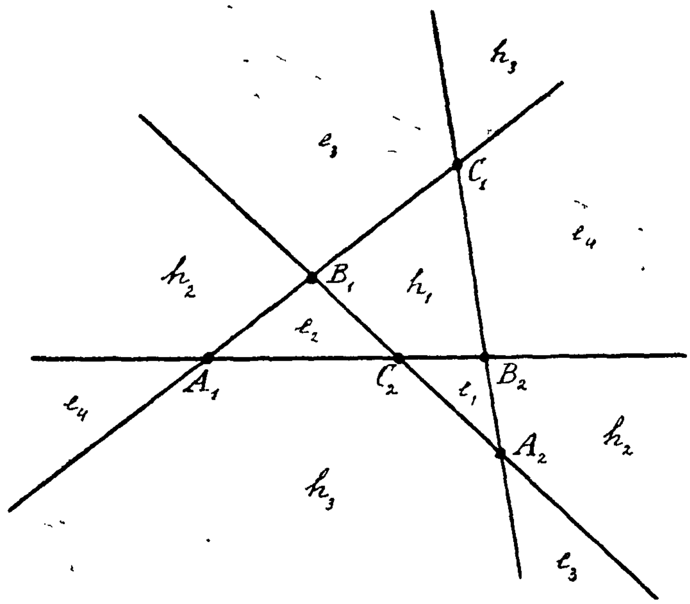

Fig. 2.

them, three pairs of conjugate points of the same system, and these do not occur on the cubic with a cusp, because through each point of such a curve only *one* tangent touching the curve elsewhere can be drawn. In this special case the locus of the cusps has broken up into the four sides of the quadrilateral each of those lines counted three times. For it is clear that an arbitrary point of the line $A_1 B_1 C_1$ e. g., as a point of contact of this line with a conic passing through A_2, B_2, C_2, represents a cusp of the linear system of cubics. We can even expect that each of the four sides must be taken into account more than one time, because each of those points instead of being an ordinary cusp is a point, where two continuing branches touch each other. And finally the remark that the sides of the quadrilateral divide the plane into four triangles e with elliptic and three quadrangles h with four hyperbolic points, so that they continue to form the separation between those two domains, forces us to bring them an odd number of times into account, namely three times because we must arrive at a compound curve c^{12}.

Some more particulars with respect to the domains e and h. The nodal tangents of the cubic (fig. 3) passing through the three pairs of points (A_1, A_1), (B_1, B_2), (C_1, C_2) and having in P a node, are the double rays of the involution of the pairs of lines connecting

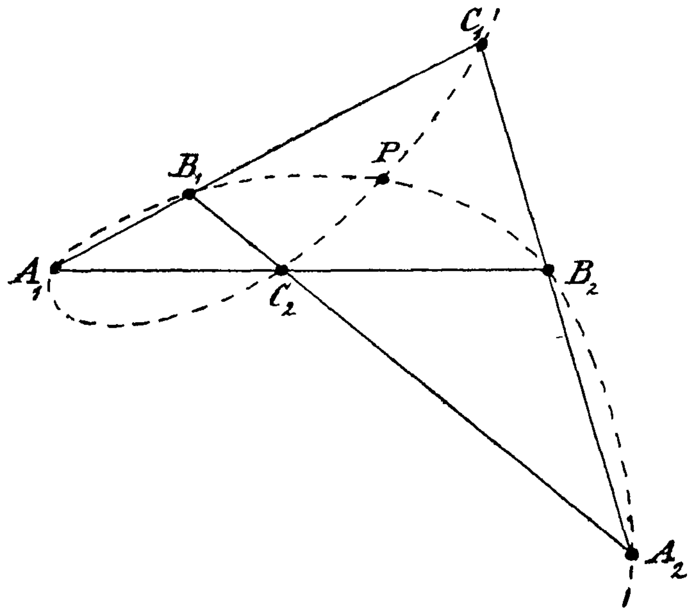

Fig. 3.

P with the three pairs of points mentioned, so also the tangents in P to the two conics of the tangential pencil with the sides of the quadrilateral as basetangents, passing through P; now, as these two conics are real or conjugate imaginary according to P lying in one of the three quadrangles h or in one of the four triangles e, what was assumed follows immediately.

To the case treated here of c^{12} broken up into four lines to be counted three times corresponds the parabolic curve of the surface S^3 with four nodes.

5. In the third place we consider still the special case of six basepoints lying on a conic, in which the linear system of cubics contains a net of curves degenerating into a conic and a right line; in this net of degenerated curves the conic is ever and again the conic c^2 through the six basepoints and the right line is an arbitrary right line of the plane.

This case can in a simple way be connected with a surface S^3 with a node O. If we project this surface out of this node O on a plane α not passing through this point, then the plane sections of the surface project as cubics passing through the six points of intersection of α with the lines of the surface passing through O; because these six lines lie on a quadratic cone, the six points of intersection with α lie on a conic. Besides, the sections with planes through O project as right lines; therefore the completing conic c^2 must evidently be regarded as the image of the node O. Of course we must here again think that c^2 corresponds point for point to the points of O^3 lying at infinite short distance from O^3; for c^2 is the section of α with the cone of the tangents to S^3 in O.

As c^2 with one of its tangents represents a curve of the linear system, this conic belongs at least twice to the locus of the cusps. Here too this locus of cusps improper with continuing branches must be accounted for three times, so that the locus proper is a curve c^6 of order six, touching c^2 in the six basepoints.

Let us suppose that c^2 is a circle and that the six basepoints on that circle (fig. 4) form the vertices of a regular hexagon, then the curve c^6 has the shape of a rosette with six leaves having the centre O' of the circle and the points at infinite distance of the diameters A_1A_4, A_2A_5, A_3A_6 as isolated points. Of the ten ovals there are four contracted to points, whilst the six remaining ones have joined into the circle of the basepoints and the curve c^6.

If we take point O' as origin and the line $O'A_1$ as x-axis of a rectangular system of coordinates, then if $O'A_1$ is unity of length we find for the equation of c^6

$$4y^2(y^2-3x^2)^2+9(x^2+y^2)^2-9(x^2+y^2)=0.$$

It is evident from this equation that the curve c^6 can really stand

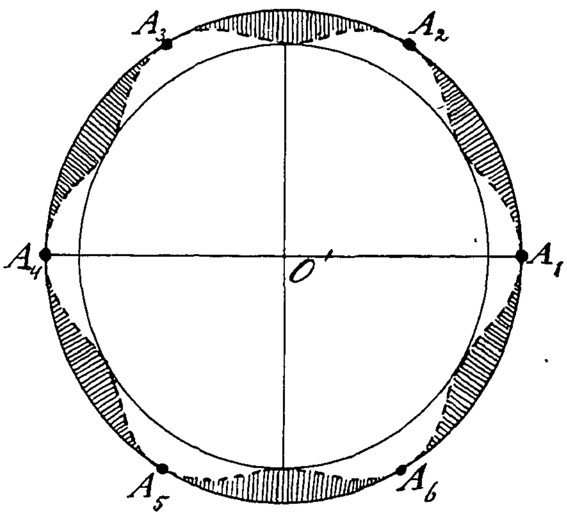

Fig. 4.

rotation of multiples of 60° round O', for then $x^2 + y^2$ and $y(y^2 - 3x^2$ are transformed into themselves.

Out of the equation

$$sin\ 3\varphi = \pm \frac{3}{2r^2} \sqrt{1 - r^2}$$

on polar coordinates it is evident that the curve c^6 (with the exception of its four isolated points) is included between the circles described out of O' with the radii 1 and $\frac{1}{2}\sqrt{3}$.

If we pass from the locus of the cusps to the parabolic curve of S^3 we must notice that the last curve has the node O of S^3 as threefold point, because c^2 has separated itself three times from the locus c^{12}. So this parabolic curve is an s^9 of order nine, a result which will presently be arrived at in an other way.

We shall give — without wishing in the least to exhaust this case of the six basepoints situated on a conic — some degenerations of the remaining curve c^6 corresponding to some definite coincidences of the basepoints.

a) The cases (2, 2, 2), (4, 2), (6). If the six basepoints coincide two by two in three points of the conic, then c^6 consists of the sides of the triangle of the basepoints counted double, originating from compound cubics with a double line; there is not a locus proper. In reality the case (2, 2, 2) of a conic touching in three points cannot occur for a genuine cubic with a cusp.

The cases (4, 2) and (6) are to be regarded as included in the preceding. By allowing two of the vertices or the three vertices of the triangle just considered to coincide we find for case (4, 2) the connecting line of the two basepoints counted four times and the tangent to the conic in the basepoint of highest multiplicity counted two times, for case (6) the tangent to the conic in the point counting for six basepoints counted six times. That there can be no locus proper in the last case ensues also from the fact, that a genuine cubic with cusp allows of no sextactic point.

b) The case (3, 3). If the six basepoints coincide three by three in two points of the conic, then c^6 consists of a part improper, the connecting line of the two points counted four times, and a part proper, a conic touching the conic of the basepoints in these points. The new conic lies *outside* the conic of the basepoints.

c) The case (1, 5). This case agrees in many respects with the preceding. We find a part improper, the tangent in the point counting for five basepoints drawn to the conic of the basepoints, and a part proper, a conic touching the conic of the basepoints in these points. The new conic lies *inside* the conic of the basepoints.

6. Of course it is possible to call forth by the curve c^{12} successively all the different special cases which can put in an appearance by the parabolic curve s^{12} of the various surfaces S^3. As this would lead us here too far, we limit ourselves to a single remark, which can eventually facilitate an analytic investigation of this idea.

According to the general results with respect to a linear system of curves c^n obtained as early as 1879 by E. Caporali the locus $c^{4(2n-3)}$ of the cusps of this system has in each r-fold basepoint of the system a $4(2r-1)$ fold point and besides $6(n-1)^2-2\Sigma(3r^2-2r+1)$ nodes C. Each of those points C is characterized by the property that each curve of the system passing through this point is touched in this point by a definite line c.

For the case under observation, $n=3$ of the cubics, the number of points C is represented by $24-6p$, when p is the number of basepoints.

If we wish to investigate analytically what peculiarity the locus of the cusps shows in a basepoint of the system, or how a line through three basepoints separates from it, then the result — and this is the remark indicated — will be independent of the fact, whether the remaining basepoints occur or not, if in the former case, that some of these basepoints appear in a real or in an imaginary condition, we assume that these points both with respect to each other and to the former basepoints have not a particular position.

With the aid of this remark we can easily find the following theorems, with which we conclude:

"Both cusps of which the fourfold point of the curve c_{12} coinciding with a basepoint A'_i seems to consist and the two cusps of the curves of the system showing in this point a cusp, coincide in cuspidal tangents, but they turn their points to opposite sides."

"If the three basepoints A'_1, A'_2, A'_3 lie on a right line l, the locus proper of the cusps reduces itself to a curve c^9 touching the line l in A'_1, A'_2, A'_3. If the three remaining basepoints exist then the points of intersection of l with the sides of the triangle having those basepoints as vertices are points of c^9".

The last case answers to that of a surface S^3 with a double point; the parabolic curve having in this doublepoint a threefold point, because l separates itself three times from c^{12}, is as has been found above already a twisted curve of order nine.

Physics. — *"An investigation of some ultra-red metallic spectra."* By W. J. H. MOLL. (Communicated by Prof. W. H. JULIUS).

(Communicated in the meeting of December 29, 1906).

Among the spectra of known elements those of the alkali-metals, by their relatively simple structure, lend themselves particularly well to an investigation of their ultra-red parts. Many observers have consequently sought for emission lines of these metals in this region.

For the first part of the ultra-red spectrum the photographic plate may be sensitised; especially LEHMANN [1]) measured in this way various lines with wave-lengths ranging to almost 1 μ. By means of the bolometer SNOW [2]) could advance to 1.5 μ.

For the further region, however, nothing was known about these spectra. COBLENTZ [3]), to be sure, was led by a series of observations in this respect, to the conclusion that the alkali-metals emit no specific radiation beyond 1.5 μ, but I had reason to doubt the validity of this conclusion.

In what follows I will briefly describe the method by which some ultra-red spectra were investigated, and the lines thus found. In an

1) H. LEHMANN. D.'s Ann. 5, 633, 1901.

2) B. W. SNOW. W.'s Ann. 47, 208, 1892.

3) W. W. COBLENTZ. Investigations of Infra-red Spectra. Carnegie Inst. Washington. 1905.