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case, it is easy to reason that PP' coincides with AB, CD or EF and so the locus proper consists of these three lines and there is no envelope proper. The part improper of the locus however consists of six conics $ABCDE$, $ABCDF$, $ABEFC$, $ABEFD$, $CDEFA$ and $CDEFB$, the part improper of the envelope of the six points A, B, C, D, E and F. The total locus is thus of order fifteen, the total envelope of class six, so that for arbitrary position of the pencils of conics this same holds for the locus proper and the envelope proper.

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Mathematics. — *"The locus of the pairs of common points of four pencils of surfaces."* By Dr. F. SCHUH. (Communicated by Prof. P. H. SCHOUTE).

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1. Given four pencils of surfaces (F_r), (F_s), (F_t) and (F_u) respectively of order r, s, t and u. The base-curves of those pencils can have common points or they can in part coincide, in consequence of which of three arbitrary surfaces of the pencils (F_s), (F_t) and (F_u) the number of points of intersection differing from the base-curves can become less than $s\,t\,u$; we call this number a, calling it b for the pencils (F_r), (F_t) and (F_u), c for the pencils (F_r) (F_s) and (F_u) and d for the pencils (F_r), (F_s) and (F_t). We now put the question:

What is the order of the surface formed by the pairs of points P and P', through which a surface of each of the four pencils is possible?

If the points P and P' do not lie on the base-curves we call the locus formed by those points the *locus proper* L on which of course still curves of points P may lie for which the corresponding point P' lies on one of the base-curves. If *one* triplet of pencils furnishes at least several points of intersection which are situated for all surfaces of those pencils on one of the base-curves, then there is a surface that *does* satisfy the question but in such a manner that if we assume P arbitrarily on this surface the point P' belonging to it is to be found on one of the base-curves; this surface we call the *part improper* of the locus, whilst both surfaces together are called the *total locus*.

2. To determine the order n of the locus proper L we find the points of intersection with an arbitrary right line l. On l we take

an arbitrary point Q_{stu} and we bring through that point surfaces F_s, F_t and F_u of the pencils (F_s), (F_t) and (F_u). Through each of the $a-1$ points of intersection of those surfaces not situated on the base-curves of those surfaces we bring a surface F_r. These $a-1$ surfaces F_r intersect the right line l together in $(a-1)r$ points Q_r, which we make to correspond to the point Q_{stu}. The coincidences of this correspondence are: 1st the points Q_{rstu} determining four surfaces which intersect one another once more in a point not lying on the base-curves, thus the n points of intersection with the surface L, 2nd the points of intersection with the surface R_{stu} belonging to the pencils (F_s), (F_t) and (F_u), the locus of the points S determining three surfaces whose tangential planes in S pass through one line.

To find the number of coincidences we have to determine the number of points Q_{stu} corresponding to an arbitrary point Q_r of l. To this end we take on l a point Q_{tu} arbitrarily and bring through it an F_t and an F_u. Through each of the b points of intersection of these surfaces with the surface F_r through Q_r (not lying on the base-curves) we bring an F_s, which b surfaces F_s intersect together the line l in bs points Q_s which we make to correspond to Q_{tu}. To find the number of points Q_{tu} corresponding to an arbitrary point Q_s of l we take Q_u arbitrarily on l, we bring through Q_s an F_s and through Q_u an F_u and through each of the c points of intersection of those surfaces with F_r an F_t, which furnish c surfaces F_t cutting l in ct points Q_t; reversely to Q_t belong du points Q_u, so that we find between the points Q_u and Q_t a (ct, du)-correspondence, of which the $ct+du$ coincidences give the points Q_{tu} belonging to the point Q_s. So between the points Q_{tu} and Q_s exists a $(bs, ct+du)$-correspondence, of which the coincidences consist of the r points of intersection of l with the surface F_r through Q_r and of the points Q_{stu} corresponding to Q_r; the number of these thus amounts to $bs+ct+du-r$.

So between the points Q_{stu} and Q_r there is an $(ar-r, bs+ct+du-r)$-correspondence with $ar+bs+ct+du-2r$ coincidences. To find out of this the number of points Q_{rstu} we must first determine the order of the surface R_{stu}.

This surface may be regarded as the surface of contact of the surfaces of the pencil (F_s) with the movable curves of intersections C_{tu} of the surfaces of the pencils (F_t) and (F_u) [1]. So the question is:

[1]) We shall call this surface the surface of contact of the three pencils meaning by this that in a point of this "surface of contact" the surfaces of the pencils, though not touching one another, admit of a common tangent.

3. *To determine the order of the surface of contact of a twofold infinite system of twisted curves and a singly infinite system of surfaces.*

To this end we shall first suppose the two systems to be arbitrary.

To determine the order of the surface of contact we count its points of intersection with an arbitrary right line l. To this end we consider the envelope E_1 of the ∞^2 tangential planes of the curves of the system in their points of intersection with l and the envelope E_2 of the ∞^1 tangential planes of the surfaces of the system in their points of intersection with l.

The common tangential planes not passing through l of both envelopes indicate by means of their points of intersection with l the points of intersection of l with the surface of contact.

In order to find the class of the envelope E_1 (formed by the tangential planes of a regulus with l as directrix) we determine the class of the cone enveloped by the tangential planes passing through an arbitrary point Q of l. If the system of curves is such that φ curves pass through an arbitrary point and ψ curves touch a given plane in a point of a given right line, the tangential planes of E_1 through Q envelope the φ tangents in Q of the curves of the system through Q, and the line l counting ψ times; for each plane through l is to be regarded ψ times as tangential plane, there being ψ curves of the system cutting l and having a tangent situated in this plane. *The envelope E_1 is thus of class $\varphi + \psi$ and has l as ψ-fold line*[1]).

To find the class of the envelope E_2 we determine the number of its tangential planes through an arbitrary point Q of l. If now the system has μ surfaces through a given point and ν surfaces touching a given right line, the tangential planes of the envelope passing through Q are the tangential planes in Q to the μ surfaces passing through Q and the tangential planes of the ν surfaces touching l. *So the envelope E_2 is of class $\mu + \nu$ with ν tangential planes through l.*

Hence both envelopes have $(\varphi+\psi)(\mu+\nu)$ common tangential planes. Each of the ν tangential planes of E_2 passing through l is however a ψ-fold tangential plane of E_1 and so it counts for ψ common tangential planes. So for the number of common tangential planes not passing through l, thus the number of points of intersection of l with the surface of contact we find:

$$(\varphi + \psi)(\mu + \nu) - \psi\nu = \varphi\nu + \psi\mu + \varphi\mu ,$$

therefore :

1) The regulus as locus of points has however line l as φ-fold line.

The surface of contact of a system (φ, ψ) *of* ∞^2 *twisted curves* [1]) *and a system* (μ, ν) *of* ∞^1 *surfaces* [2]) *is of order* $\varphi\nu + \psi\mu + \varphi\mu$ [3]).

4. To determine the order of the surface of contact[4]) of the systems (μ_1, ν_1), (μ_2, ν_2) and (μ_3, ν_3) each of ∞^1 surfaces, we regard the system (φ, ψ) of the curves of intersection of the systems (μ_1, ν_1) and (μ_2, ν_2). Of these curves of intersection $\mu_1\mu_2$ pass through a given point, so $\varphi = \mu_1\mu_2$. The ψ points, where the curves of intersection touch a given plane in a point of a given right line, are the points of intersection of that given line with the curve of contact of the systems (μ_1, ν_1)[5]) and (μ_2, ν_2) of plane curves, according to which the given plane intersects the systems of surfaces (μ_1, ν_1) and (μ_2, ν_2). This curve of contact is of order $\mu_1\nu_2 + \mu_2\nu_1 + \mu_1\mu_2$, thus:

$$\psi = \mu_1\nu_2 + \mu_2\nu_1 + \mu_1\mu_2.$$

The surface of contact to be found is thus the surface of contact of a system $(\mu_1\mu_2, \mu_1\nu_2 + \mu_2\nu_1 + \mu_1\mu_2)$ of ∞^2 twisted curves and a system (μ_3, ν_3) of ∞^1 surfaces, so that we find:

The surface of contact of three systems (μ_1, ν_1), (μ_2, ν_2) *and* (μ_3, ν_3) *of* ∞^1 *surfaces is of order*

$$\mu_2\mu_3\nu_1 + \mu_3\mu_1\nu_2 + \mu_1\mu_2\nu_3 + 2\mu_1\mu_2\mu_3.$$

If the three systems are the pencils (F_s), (F_t) and (F_u) we have

$$\mu_1 = \mu_2 = \mu_3 = 1,$$

$$\nu_1 = 2(s-1) \quad , \quad \nu_2 = 2(t-1) \quad , \quad \nu_3 = 2(u-1).$$

So we find:

The surface of contact F_{stu} *of the three pencils of surfaces* (F_s), (F_t) *and* (F_u) *is of order*

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[1]) System with φ curves through a given point and ψ curves cutting a given line and touching in the point of intersection a given plane through that line.

[2]) System with μ surfaces through a given point and ν surfaces touching a given right line.

[3]) This result is also immediately deducible from the SCHUBERT formula

$$xp^2 = p'^3 \cdot G + p'g'_e \cdot p^2g_e + p'^3 \cdot p^2g_e$$

(Kalkul der abzählenden Geometrie, formula 13, page 292) for the number of common elements with a point lying on a given line of a system Σ' of ∞^3 and a system Σ of ∞^1 right lines with a point on it. If we take for Σ' the tangents with point of contact of the system of curves (φ, ψ) and for Σ the tangents with point of contact of the system of surfaces (μ, ν), then

$$p'^3 = \varphi \quad , \quad p'g'_e = \psi \quad , \quad G = \nu \quad , \quad p^2g_e = \mu ,$$

whilst xp^2 is the order of the surface of contact.

[4]) Locus of the points, where the surfaces of the three systems have a common tangent.

[5]) System of ∞^1 curves of which μ_1 pass through a given point and ν_1 touch a given right line.

$$2(s+t+u-2).$$

5. To return to the question which gave rise to the preceding considerations we find for the number of points Q_{rstu} on the arbitrary line l, which are the points of intersection of l with the locus proper L:

$$ar + bs + ct + du - 2r - 2(s+t+u-2) =$$
$$= ar + bs + ct + du - 2(r+s+t+u) + 4.$$

So we find:

The locus L of the pairs consisting of two movable points common to a surface out of each of the pencils (F_r), (F_s), (F_t) *and* (F_u) *of orders r, s, t and u, and not lying on the base-curves, is a surface of order*

$$ar + bs + ct + du - 2(r+s+t+u) + 4.$$

Here a is the number of points of intersection not necessarily situated on the base-curves of the pencils (F_s), (F_t) *and* (F_u); *b the analogous number for the pencils* (F_r), (F_t) *and* (F_u), *etc.*

6. It the pencils have an arbitrary situation with respect to each other, then $a = stu$, etc., so that then the order of the locus becomes

$$4(rstu+1) - 2(r+s+t+u).$$

That order is lowered when three of the base-curves have a common point or two of the base-curves have a common part, which lowering of the order can be explained by separation as long as the total locus is definite, i. e. as long as the four base-curves have no common point and no triplet of base-curves have a common part. For, if A_{rstu} is a common point of four base-curves then the surfaces of the four pencils passing through an entirely arbitrary point P have another second point in common, namely A_{rstu}; if B_{stu} is a curve forming part of the base-curves B_s, B_t and B_u of the pencils (F_s), (F_t) and (F_u), then the surfaces of the pencils passing through an arbitrary point P have moreover the points of intersection in common of B_{stu} with the surface F_r through P; so in both cases the arbitrary point P belongs to the total locus.

If the basecurves B_s, B_t and B_u have a common point A_{stu} then on account of that point the number a is diminished by unity without having any influence on b, c and d. The order of L is thus lowered by r on account of it, which is immediately explained by the fact that *the surface* F_r *passing through* A_{stu} *separates itself from the locus.*

If the base-curves B_t and B_u have a curve B_{tu} in common of which for convenience we suppose that it does not intersect the base-curves B_r and B_s, this B_{tu} has no influence on c and d, whilst a is lowered with sm and b with rm, where m represents the order of the curve B_{tu}; for, when F_s, F_t and F_u are three arbitrary surfaces always sm points of intersection lie on B_{tu}. The order of L is thus lowered with $2rsm$ by B_{tu}. This can be explained by the fact, that *the locus of the curves of intersection C_{rs} of surfaces F_r and F_s passing through a selfsame point of B_{tu}* [1]) *separates itself from the locus of P and P'.* That the locus of those curves of intersection is really of order $2rsm$ is easily evident from the points of intersection with an arbitrary line l. We can bring through an arbitrary point Q_r of l an F_r cutting B_{tu} in rm points, through each of those points of intersection we bring an F_s, which rm surfaces F_s cut the right line l in rsm points Q_s. To Q_r correspond rsm points Q_s and reversely. The $2rsm$ coincidences are the points of intersection of l with the locus of the curves of intersection C_{rs}.

7. The base-curves B_r, B_s, B_t and B_u of the pencils are morefold curves of the surface L. If A_r is a point of B_r but not of the other base-curves, then A_r is an $(a-1)$-fold point of L. For, the surfaces F_s, F_t and F_u through A_r intersect one another in $a-1$ points, not lying on the base-curves, each of which points furnishes together with A_r a pair of points satisfying the question. Each point of B_r is thus an $(a-1)$-fold point, i. o. w. *B_r is $(a-1)$-fold curve of the surface L.*

Let A_{rs} be a point of intersection of the base-curves B_r and B_s, but not a point of B_t and B_u. An arbitrary point P of the curve of intersection C_{tu} of the surfaces F_t and F_u through A_{rs} furnishes now together with A_{rs} a pair of points PP' satisfying the question properly, as A_{rs} is for each triplet of pencils a movable point of intersection not lying on the base-curves. If we let P describe the curve C_{tu}, then the tangent l_{rs} in A_{rs} to the curve of intersection of the surfaces F_r and F_s through P describes the cone of contact of L in the conic point A_{rs}. The tangents m_r and m_s in A_{rs} to B_r and B_s are $(a-1)$- resp. $(b-1)$-fold edges of the cone. This cone is cut by the plane through m_r and m_s only according to the line m_r counting $(a-1)$-times and the line m_s counting $(b-1)$-times, as another line l_{rs} lying in this plane would determine two surfaces

[1]) If B_{tu} cuts the curve B_s in a point A_{stu}, then the surface F_r passing through A_{stu} separates itself from the locus of the curves of intersection C_{rs}.

F_r and F_s touching each other in A_{rs}, whose curve of intersection, however, does not cut the curve C_{tu}. *The tangential cone of L in A_{rs} is thus of order $a+b-2$* [1]).

Let $A_{rs}^{(1)}$ be a point of a common part B_{rs} of the base-curves B_r and B_s but not a point of B_t and B_u. We get a pair of points PP' with a point P' coinciding with $A_{rs}^{(1)}$ when the surfaces F_r and F_s have in $A_{rs}^{(1)}$ a common tangential plane V_{rs} and pass through a selfsame point P of the curve of intersection C_{tu} of the surfaces F_t and F_u through $A_{rs}^{(1)}$. If we let P describe the curve C_{tu}, then on account of that between the planes V_r and V_s, touching in $A_{rs}^{(1)}$ the surfaces F_r and F_s through P, a correspondence is arranged, where to V_r correspond $b-1$ planes V_s and to V_s correspond $a-1$ planes V_r. One of the $a+b-2$ planes of coincidences is the plane through the tangents in $A_{rs}^{(1)}$ to B_{rs} and C_{tu}; this plane furnishes no plane V_{rs}. The remaining $a+b-3$ planes of coincidence are planes V_{rs} and indicate the tangential planes in $A_{rs}^{(1)}$ to the surface L. *So B_{rs} is an $(a+b-3)$-fold curve of L.*

8. Let us then consider a common point A_{rst} of the base-curves B_r, B_s and B_t. We get a pair of points PP' with a point P' coinciding with A_{rst}, when the tangential planes in A_{rst} to F_r, F_s and F_t pass through one line l_{rst} and these surfaces intersect one another again in a point P of the surface F_u passing through A_{rst}. There are ∞^1 such lines l_{rst}, forming the tangential cone of L in point A_{rst}. The tangents m_r, m_s and m_t in A_{rst} to B_r, B_s and B_t are $(a-1)$-, $(b-1)$- and $(c-1)$-fold edges of that cone. So the plane through m_r and m_s furnishes $a+b-2$ lines of intersection with the cone coinciding with m_r and m_s. Moreover $c-2$ other lines l_{rst} lie in this plane. For, the surfaces F_r and F_s touching this plane intersect F_u in $c-2$ points not lying on the base-curves; the surfaces F_t through those points intersect the plane through m_r and m_s according to curves whose tangents in A_{rst} are the mentioned

[1]) The order of this cone can also be found out of the number of lines of intersection with an arbitrary plane ε through A_{rs}. If l_r and l_s are the lines of intersection of ε with the tangential planes in A_{rs} to the surfaces F_r and F_s through P, then to l_r correspond $b-1$ lines l_s and to l_s correspond $a-1$ lines l_r, so that in the plane ε lie $a+b-2$ lines l_{rs}.

lines l_{rst}. *So the tangential cone of L in A_{rst} is of order $a+b+c-4$* [1]).

A point of intersection $A_{rst}^{(1)}$ of B_r with a common part B_{st} of the base-curves B_s and B_t is a conic point of L, the tangential cone of which is formed as in the previous case by ∞^1 lines l_{rst}. The tangents m_r and m_{st} in $A_{rst}^{(1)}$ to B_r and B_{st} are $(a-1)$ and $(b+c-3)$-fold edges of that cone. As no other lines l_{rst} lie in the plane through m_r and m_{st}, it is evident *that the tangential cone of L in $A_{rst}^{(1)}$ is likewise of order $a+b+c-4$* [1]).

Let $A_{rst}^{(2)}$ be a point of a common part B_{rst} of the base-curves B_r, B_s and B_t. The point P' of the pair of points PP' coincides with $A_{rst}^{(2)}$ when the surfaces F_r, F_s and F_t have in $A_{rst}^{(2)}$ the same tangential plane V_{rst} and cut one another in another point P of the surface F_u through $A_{rst}^{(2)}$. If we now consider an F_r and an F_s having in $A_{rst}^{(2)}$ the same tangential plane V_{rs} and if we consider through each of the $c-1$ points of intersection of F_r, F_s and F_u not lying on the base-curves an F_t of which we indicate the tangential plane in $A_{rst}^{(2)}$ by V_t then to V_{rs} correspond $c-1$ planes V_t and to V_t correspond $a+b-1$ planes V_{rs} (as for given V_t a (b, a)-correspondence exists between V_r and V_s of which V_t is one of the planes of coincidence). Among the $a+b+c-2$ planes of coincidence $V_{rs}\,V_t$ there are however *three* which give no plane V_{rst}, namely the planes V_{rs}, for which the corresponding surfaces F_r and F_s furnish with F_u three points of intersection coinciding with $A_{rst}^{(2)}$. For this is necessary that F_u touches in $A_{rst}^{(2)}$ the movable intersection of F_r and F_s. Now the tangents of those intersections for all surfaces F_r and F_s touching each other in $A_{rst}^{(2)}$ form a cubic cone having for double edge the tangent m_{rst} to B_{rst} in point $A_{rst}^{(2)}$ [2]). This cone is cut by the tangential plane in $A_{rst}^{(2)}$ to F_u according to three lines, furnishing with m_{rst} planes V_{rs} which are planes of coincidence

[1]) This order can also be determined out of the number of lines l_{rst} in a plane ε passing through A_{rst}. In this plane we find a $(c-1,\ a+b-2)$-correspondence between lines l_{rs} and lines l_t of which however the line of intersection of ε with the tangential plane in A_{rst} to F_u is a line of coincidence, but no line l_{rst}.

[2]) This is immediately evident if we take for (F_r) a pencil of planes and for (F_s) a pencil of quadratic surfaces all passing through the axis B_r of the pencil of planes. The cone under consideration then becomes the cone of the generatrices of the quadratic surfaces passing through a given point of B_r. We can easily convince ourselves that the same result holds for arbitrary pencils of surfaces.

of V_{rs} and V_t, but not planes V_{rst}. So there are $a+b+c-5$ planes V_{rst}, which are the tangential planes of L in the point $A^{(2)}_{rst}$, i. o. w. *B_{rst} is $(a+b+c-5)$-fold curve of surface L.*

9. We then consider a common point A_{rstu} of the four base-curves. We get a pair of points PP' with point P' coinciding with A_{rstu} when F_r, F_s, F_t and F_u have in A_{rstu} a common tangent l_{rstu} and all pass once again through a selfsame point P. The ∞^1 lines l_{rstu} form the tangential cone of L in A_{rstu}. To determine the number of lines l_{rstu} in an arbitrary plane ε through A_{rstu} we take in this plane an arbitrary line l_{rst} through A_{rstu} and we bring through the $d-1$ points of intersection (not lying on the base-curves) of the surfaces F_r, F_s and F_t touching l_{rst} the surfaces F_u, whose tangential planes in A_{rstu} cut the plane ε according to lines, which we shall call l_u. To l_{rst} now correspond $d-1$ lines l_u and to l_u correspond $a+b+c-2$ lines l_{rst}, as there exists between l_{rs} and l_t when l_u is given a $(c, a+b)$-correspondence, of which l_u and the line of intersection of ε with the plane through the tangents in A_{rstu} to B_r and B_s are lines of coincidence, but not lines l_{rst}. So there are $a+b+c+d-3$ lines of coincidence l_{rst} l_u of which however *three* are not lines l_{rstu}. The common tangents in A_{rstu} of the surfaces F_r, F_s and F_t possessing three points of intersection coinciding with A_{rstu} and where therefore the intersection of two of those surfaces shows a contact of order two to the third, form namely a cubic cone [1]) of which the lines of intersection with ε are lines of coincidence but not lines l_{rstu}. So in ε lie $a+b+c+d-6$ lines l_{rstu}, i. o. w. *the tangential cone of L in A_{rstu} is of order $a+b+c+d-6$* [2]).

1) This is again evident when taking for (F_r) and (F_s) pencils of planes with coplanar axes B_r and B_s and for (F_t) a pencil of quadratic surfaces passing through a line containing the point of intersection S of B_r and B_s. The line of intersection of the planes F_r and F_s shows only then a contact of order two to F_t when that line of intersection lies entirely on F_t, so that the cone under consideration becomes again the cone of the generatrices of the quadratic surfaces passing through S.

2) That order can also be found out of the lines of intersection with the plane V_{rs} through the tangents m_r and m_s in A_{rstu} to B_r and B_s. Those lines of intersection are: the line m_r, counting $(a-1)$-times, the line m_s counting $(b-1)$-times and $c+d-4$ other lines. This last amount we find by drawing in plane V_{rs} an arbitrary line l_t through A_{rstu}. The surface F_t touching l_t cuts the surfaces F_r and F_s touching V_{rs} in $d-1$ points (not lying on the base-curves) through which points we bring surfaces F_u whose tangential planes in A_{rstu} cut the plane V_{rs} according to lines to be called l_u. Between the lines l_t and l_u we now have a $(d-1, c-1)$-correspondence of which the nodal tangents in A_{rstu} of the intersection of the surfaces F_r and F_s touching V_{rs} are lines of coincidence. The remaining $c+d-4$ lines of coincidence are lines l_{rstu}.

The preceding considerations hold invariably for a point $A^{(1)}_{rstu}$ lying on the base-curves B_r and B_s and the common part B_{tu} of the base-curves B_t and B_u [1]).

In a point of intersection $A^{(2)}_{rstu}$ of B_{rs} and B_{tu} the tangential cone is likewise of order $a+b+c+d-6$ as that cone has the tangents m_{rs} and m_{tu} to B_{rs} and B_{tu} as $(a+b-3)$ and $(c+d-3)$ fold edges, whilst in the plane through m_{rs} and m_{tu} no other right lines l_{rstu} are lying.

A point of intersection $A^{(3)}_{rstu}$ of B_r and B_{stu} is also a $(a+b+c+d-6)$ fold point of L as m_r and m_{stu} are $(a-1)$- and $(b+c+d-5)$-fold edges of the tangential cone and the only lines of intersection of that cone with the plane through m_r and m_{stu}.

If finally $A^{(4)}_{rstu}$ is a point of a common part B_{rstu} of the four base-curves, then the point P' of the pair of points PP' coincides with $A^{(4)}_{rstu}$ when the surfaces F_r, F_s, F_t and F_u have in $A^{(4)}_{rstu}$ the same tangential plane V_{rstu} and all pass through a same point P. Let us now assume an arbitrary plane V_{rst} passing through the tangent m_{rstu} in $A^{(4)}_{rstu}$ to B_{rstu}. The surfaces F_r, F_s and F_t touching this plane in $A^{(4)}_{rstu}$ cut one another in $d-1$ points P, through which we bring surfaces F_u, of which we call the tangential planes in $A^{(4)}_{rstu}$ V_u. Thus we obtain a correspondence, where to V_{rst} correspond $d-1$ planes V_u and reversely to V_u correspond $a+b+c-1$ planes V_{rst}; for when V_u is given there is between V_{rs} and V_t a $(c, a+b)$ correspondence, of which V_u is plane of coincidence, but not a plane V_{rst}. So there are $a+b+c+d-2$ planes of coincidence $V_{rst}V_u$, of which however *five* are not planes V_{rstu}. These are namely the tangential planes of the surfaces F_r, F_s and F_t of which one more point of intersection coincides with $A^{(4)}_{rstu}$ which

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1) It is also easy to see from the lines of intersection with the plane V_{stu} through the tangents m_s and m_{tu} to B_s and B_{tu} that the tangential cone in $A^{(1)}_{rstu}$ is of order $a+b+c+d-6$. The line m_s counts for $b-1$ lines of intersection, the line m_{tu} for $c+d-3$. Further, the surfaces F_s, F_t and F_u touching V_{stu} cut one another in $a-2$ points not lying on the base-curves; through those points we bring surfaces F_r, whose tangential planes in $A^{(1)}_{rstu}$ cut the plane V_{stu} along to lines which lie on the tangential cone.

occurs five times [1]). So there remain $a+b+c+d-7$ planes V_{rstu} which are the tangential planes of L in the point $A^{(4)}_{rstu}$, so that B_{rstu} is a $(a+b+c+d-7)$ fold curve of L.

10. So we find:
Of the locus proper L of the pairs of points P and P' the base-curve B_r of the pencil (F_r) is $(a-1)$-fold curve, the common part B_{rs} of the base-curves B_r and B_s is $(a+b-3)$-fold curve, the common part B_{rst} of the base-curves B_r, B_s and B_t is $(a+b+c-5)$ fold curve and the common part B_{rstu} of the four base-curves is $(a+b+c+d-7)$-fold curve. The points of intersection of the base-curves are conic points of L, namely a point of intersection of B_r and B_s is $(a+b-2)$-fold point, a point of intersection of B_r, B_s and B_t or of B_r and B_{st} is $(a+b+c-4)$-fold point and a point of intersection of B_r, B_s, B_t and B_u or of B_r, B_s and B_{tu} or of B_{rs} and B_{tu} or of B_r and B_{stu} is $(a+b+c+d-6)$-fold point. [2])

11. The base-curves of the pencils are not the only singular curves of the surface L. There are namely ∞^1 triplets of points lying on a surface of each of the pencils. These triplets of points form a double curve of L. If P, P', P'' is such a triplet and if $P1$ and $P2$ are the sheets through P of the surface, then the sheets $P'1$ and $P''2$ correspond to them. Through P' passes another sheet $P'3$ and through P'' a sheet $P''3$ which sheets correspond mutually. The pair of points not lying on the base-curves is movable along the sheets $P1$, $P'1$, along the sheets $P2$, $P''2$ and along the sheets $P'3$, $P''3$, on the base-curve a third point then joins the pair.

Further there is still a finite number of quadruples of points,

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1) The number *five* is found in the following way. The tangents of the movable intersections of surfaces F_s and F_t touching each other in $A^{(4)}_{rstu}$ form a cubic cone having the tangent m_{rstu} to B_{rstu} as double line. Such an intersection shows to the surface F_r a contact of order two when it touches the movable intersection of F_r and F_t, so if its tangent in $A^{(4)}_{rstu}$ lies on the cubic cone belonging to the pencils (F_r) and (F_t). As this last cone has also m_{rstu} as double edge, both cones have $9-4=5$ lines of intersection differing from m_{rstu} which connected with m_{rstu} furnish the five planes under consideration.

2) If the total locus is not indefinite, i. o. w. if there is no point common to the four base-curves then B_r is a $(stu-1)$ fold curve and B_{rs} a $(stu+rtu-2)$ fold curve of the total locus whilst a point of intersection of B_r and B_s is a $(stu+rtu-2)$-fold point and a point of intersection of B_r, B_s and B_t or of B_r and B_{st} a $(stu+rtu+rsu-3)$-fold point of it.

through which passes a surface out of each of the pencils. Through the points P, P', P'' and P''' of such a quadruple pass three sheets of the surface L and three branches of the double curve. The 12 branches of the double curve through those four points we can call $P1, P2, P3, P'1, P'2, P'4, P''1, P''3, P''4, P'''2, P'''3, P'''4$, in such a way that the triplet of points is movable along the branches $P1, P'1, P''1$, along $P2, P'2, P'''2$, along $P3, P''3, P'''3$ and along $P'4, P''4, P'''4$. If the sheet of L passing through $P1$ and $P2$ is called $P12$, then the corresponding sheets (i. e. sheets along which the pair of points not lying on the double curve is movable) are $P12$ and $P'12, P13$ and $P''13$, etc.

Geophysics. — "*Current-measurements at various depths in the North Sea.*" (First communication). By Prof. C. H. Wind, Lt^{t}. A. F. H. Dalhuisen and Dr. W. E. Ringer.

In the year 1904 accurate measurements of the currents in the North Sea [1]) were started by the naval lieutenant A. M. van Roosendaal, at the time detached to the "Rijksinstituut voor het Onderzoek der Zee", having been proposed and guided by the Dutch delegates to the International Council for the Study of the Sea.

By him four apparatus were put to the test, viz. 2 specimens of the current-meter of Pettersson [2]), one of that of Nansen [3]) and one of that of Ekman [4]), all destined to determine the direction and the velocity of the current at every depth.

The experiments were partly made on the light-ship "Haaks", where Dr. J. P. van der Stok, the Marine Superintendent of the Kon. Nederl. Meteorologisch Instituut, also took part in them. Other experiments were made in the harbour of Nieuwediep and further, from the research-steamer "Wodan", in the open North Sea at a station (H2) of the Dutch seasonal cruises [5]), situated at Lat. 53°44′ N. and Long. 4°28′ E.

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1) Cons. Perm. Intern. p. l'expl. de la mer, Publications de circonstance No. 26: A. M. van Roosendaal und C. H. Wind, Prüfung von Strommessern und Strommessungsversuche in der Nordsee. Copenhague, 1905.

2) Publ. de circ. No. 25.

3) „ „ „ No. 34.

4) „ „ „ No. 24.

5) Quarterly cruises of the countries taking part in the international study of the sea, along fixed routes, observations being made at definite points or "stations".