that the deviations for a part considerably exceed the limits of accurateness of the statements.

It should be observed that the charts refer to currents near the surface, whereas the values of the table derived from our observations refer to a depth of 5 M.

Finally we may mention that the observations at station H2 up till now have been continued in the same way, that is to say, they are still made every quarter of a year, as far as possible, during 24 hours. Moreover, owing to the kind co-operation of His Excellency the Minister of Marine, a current-meter of PETTERSSON has been placed on the lightship "Noord-Hinder", with which since November 1906 daily, in so far as the state of the weather permits, with intervals of three hours, measurements at various depths are made by the ordinary staff of the lightship. The lists of observation are forwarded to the "Rijksinstituut" and promise to yield important material, especially for the inquiry into the way in which the tidal and residual currents differ in layers of different depth.

Mathematics. — *"The locus of the pairs of common points of $n+1$ pencils of $(n-1)$-dimensional varieties in a space of n dimensions."* By Dr. F. SCHUH.

(Communicated by Prof. P. H. SCHOUTE).

1. Let (V_i) $(i=1, 2, \ldots, n+1)$ be $n+1$ pencils of $(n-1)$-dimensional varieties in the space of operation Sp^n of n dimensions and let r_i be the order of the varieties V_i of the pencil (V_i). Let moreover a_i be the number of points of intersection of the n varieties $V_1, V_2, \ldots, V_{i-1}, V_{i+1}, V_{i+2}, \ldots, V_{n+1}$ not of necessity lying in the base-varieties.

When considering the locus of pairs of points P, P' through which a variety of each of the pencils passes we have exclusively such pairs in view of which neither of the two points lies of necessity on a base-variety of one of the pencils and we call the locus thus arrived at the *locus proper* L.

We determine the order of L out of its points of intersection with an arbitrary right line l. To this end we take on l an arbitrary point $Q_{12\ldots n}$ and we bring though it varieties $V_1, V_2, V_3, \ldots, V_n$, having $a_{n+1}-1$ points of intersection not lying on $Q_{12\ldots n}$ and the base-varieties. Through each of those points we bring a V_{n+1} and arrive in this way at $a_{n+1}-1$ varieties V_{n+1} intersecting together line l in $(a_{n+1}-1)\, r_{n+1}$ points Q_{n+1}. So to $Q_{12\ldots n}$ correspond $(a_{n+1}-1) r_{n+1}$ points Q_{n+1}.

To find reversely how many points $Q_{12\ldots n}$ correspond to Q_{n+1} we take arbitrarily on l the points Q_{i+1}, Q_{i+2}, $Q_{i+3}, \ldots, Q_{n+1}$ and we bring through those points respectively a $V_{i+1}, V_{i+2}, V_{i+3}, \ldots, V_{n+1}$. We now put the question how many points $Q_{123\ldots i}$ lie on l in such a way that the varieties mentioned $V_{i+1}, V_{i+2}, .., V_{n+1}$ and the varieties $V_1, V_2, .., V_i$ passing through $Q_{123\ldots i}$ have a common point not lying on the base-varieties. For $i < n$ the answer is: $a_1 r_1 + a_2 r_2 + \ldots + a_i r_i$.

To prove this we begin by noticing that the correctness is immediately evident for $i = 1$. If we now assume the correctness for $i = j$, we have only to show that the formula also holds for $i = j + 1$. Given the points Q_{j+2}, $Q_{j+3}, \ldots, Q_{n+1}$. To determine the number of points $Q_{123\ldots j+1}$ we take on l an arbitrary point $Q_{123\ldots j}$, we bring through it varieties $V_1, V_2, \ldots, V_j$ and then through each of the a_{j+1} points of intersection (not lying on the base-varieties) of these $V_1, V_2, \ldots, V_j$ and the varieties V_{j+2}, $V_{j+3}, \ldots, V_{n+1}$ resp. passing through $Q_{j+2}, Q_{j+3}, \ldots, Q_{n+1}$ we bring a variety V_{j+1}; these a_{j+1} varieties V_{j+1} cut l in $a_{j+1}\, r_{j+1}$ points Q_{j+1}. So to $Q_{123\ldots j}$ correspond $a_{j+1}\, r_{j+1}$ points Q_{j+1} and (according to the supposition that the formula holds for $i = j$) reversely to Q_{j+1} correspond $a_1 r_1 + a_2 r_2 + \ldots + a_j r_j$ points $Q_{123\ldots j}$. So there are $a_1 r_1 + a_2 r_2 + \ldots + a_j r_j + a_{j+1} r_{j+1}$ coincidences $Q_{123\ldots j}\, Q_{j+1}$; these are the points $Q_{12\ldots j\, j+1}$ belonging to the given points Q_{j+2}, $Q_{j+3}, \ldots, Q_{n+1}$; in this way the correctness of the formula has been indicated for $i = j + 1$.

When asking after the number of points $Q_{12\ldots n}$ corresponding to Q_{n+1} we have $i = n$, so that the formula furnishes $a_1 r_1 + a_2 r_2 + \ldots + a_n r_n$ for it. This number must however still be diminished by r_{n+1}, as each of the points of intersection of l with the V_{n+1} passing through Q_{n+1} is a point of coincidence $Q_{123\ldots n-1}\, Q_n$ but not one of the indicated points $Q_{12\ldots n}$.

So on l there exists between the points $Q_{12\ldots n}$ and Q_{n+1} an $(a_{n+1} r_{n+1} - r_{n+1},\, a_1 r_1 + a_2 r_2 + \ldots + a_n r_n - r_{n+1})$ correspondence. The $a_1 r_1 + a_2 r_2 + \ldots + a_{n+1} r_{n+1} - 2 r_{n+1}$ coincidences are the points of intersection of l with the locus L to be found and the points of intersection of l with the $(n-1)$-dimensional variety of contact $R\,V_{12\ldots n}$ of the pencils $(V_1), (V_2), \ldots, (V_n)$; we understand by that *variety of contact* the locus of the points, where the varieties V_1, $V_2, \ldots, V_n$ passing through them have a common tangent, so where the $(n-1)$-dimensional tangential spaces of those varieties cut each other according to a line.

2. To determine the order of $RV_{12\ldots n}$ we must observe that $RV_{12\ldots n}$ is the locus of the points of contact of the varieties V_n with the curves of intersection $C_{12\ldots n-1}$ of the varieties $V_1, V_2, \ldots, V_{n-1}$. So the question has been reduced to that of the order of the variety of contact of a system of ∞^1 $(n-1)$-dimensional varieties and a system of ∞^{n-1} curves. That order can be determined out of the points of intersection with an arbitrary line l.

In a point of intersection of l with a variety of the system we bring the $(n-1)$-dimensional tangential space Sp^{n-1} and in a point of intersection of l with a curve of the system the ∞^{n-2} tangential spaces Sp^{n-1}. If we act in the same way with all varieties and curves of both systems, then the tangential spaces of the varieties furnish an *1-dimensional envelope* E_1 (i. e. a curve) of *class* $\mu + \nu$ (as is evident out of its osculating spaces Sp^{n-1} through an arbitrary point of l) *with* ν *osculating spaces* Sp^{n-1} *passing through* l; here μ is the number of varieties of the system passing through an arbitrary point, and ν that of the varieties touching an arbitrary right line. The tangential spaces of the curves in the points of intersection with l have an $(n-1)$*-dimensional envelope* E_2 *of class* $\varphi + \psi$ *with* l *as* ψ*-fold line,* where φ is the number of curves of the system passing through an arbitrary point and ψ that of the curves touching an arbitrary space Sp^{n-1} in a point of a given right line of that space; for, if we bring through a point Q of l an arbitrary Sp^{n-2}, then each of the φ curves of the system passing through Q furnishes a tangential space Sp^{n-1} passing through this Sp^{n-2} whilst the space Sp^{n-1} determined by l and Sp^{n-2} (just as every other Sp^{n-1} passing through l) is ψ times tangential space of the envelope.

Both envelopes have thus $(\mu + \nu)(\varphi + \psi)$ common tangential spaces Sp^{n-1}. Each of the ν osculating spaces Sp^{n-1} of E_1 passing through l is a ψ-fold tangential space of E_2, so it counts for ψ common tangential spaces; so that $\mu\varphi + \mu\psi + \nu\varphi$ common tangential spaces not passing through l are left; these indicate by their points of intersection with l the points of intersection of l with the variety of contact, so we find:

The $(n-1)$*-dimensional variety of contact of an* ∞^1 *system of* $(n-1)$*-dimensional varieties of which* μ *pass through a given point and* ν *touch a given right line, and an* ∞^{n-1} *system of curves of which* φ *pass through a given point and* ψ *touch a given space* Sp^{n-1} *in a point of a given right line of that space, is of order*

$$\mu\psi + \nu\varphi + \mu\varphi.$$

3. With the aid of this result it is easy to determine the order

of the variety of contact (locus of the points with common tangent) of n simple infinite systems (μ_1, ν_1), (μ_2, ν_2), ..., (μ_n, ν_n) of $(n—1)$-dimensional varieties.

This order is

$$\mu_1 \mu_2 \dots \mu_n \left(\frac{\nu_1}{\mu_1} + \frac{\nu_2}{\mu_2} + \dots + \frac{\nu_n}{\mu_n} + n - 1 \right),$$

as can be shown by complete induction. The formula holds for $n = 2$. We assume the correctness of the formula for $n = i$ and out of this we must find the correctness for $n = i + 1$.

The variety of contact for $i + 1$ systems in Sp^{i+1} is the variety of contact of the system of varieties (μ_1, ν_1) and the system of curves formed by the intersections of the i remaining systems of varieties. So we have:

$$\mu = \mu_1 \quad , \quad \nu = \nu_1 \quad , \quad \varphi = \mu_2 \mu_3 \dots \mu_{i+1}.$$

The points of contact of the curves of the system with a given space Sp^i form the $(i—1)$-dimensional variety of contact of the sections of Sp^i with the systems (μ_2, ν_2), (μ_3, ν_3), ..., (μ_{i+1}, ν_{i+1}); these sections are likewise systems (μ_2, ν_2), ..., (μ_{i+1}, ν_{i+1}), but of $(i—1)$-dimensional varieties. The variety of contact mentioned is according to supposition of order

$$\mu_2 \mu_3 \dots \mu_{i+1} \left(\frac{\nu_2}{\mu_2} + \frac{\nu_3}{\mu_3} + \dots + \frac{\nu_{i+1}}{\mu_{i+1}} + i - 1 \right).$$

The points of intersection of that variety of contact with a right line l of Sp^i being the points of l in which Sp^i is touched by curves of the system, we have:

$$\psi = \mu_2 \mu_3 \dots \mu_{i+1} \left(\frac{\nu_2}{\mu_2} + \frac{\nu_3}{\mu_3} + \dots + \frac{\nu_{i+1}}{\mu_{i+1}} + i - 1 \right).$$

Thus according to the formula $\mu\psi + \nu\varphi + \mu\varphi$ the order of the i-dimensional variety of contact of the $i + 1$ systems of varieties becomes

$$\mu_1 \mu_2 \dots \mu_{i+1} \left(\frac{\nu_1}{\mu_1} + \frac{\nu_2}{\mu_2} + \dots + \frac{\nu_{i+1}}{\mu_{i+1}} + i \right),$$

by which the correctness of the same formula for $n = i + 1$ has been demonstrated. So we find:

For n ∞^1 systems (μ_1, ν_1), (μ_2, ν_2), ... , (μ_n, ν_n) of $(n—1)$-dimensional varieties the locus of the points where the varieties of the systems passing through it have a common tangent is an $(n—1)$-dimensional variety (variety of contact) of order

$$\mu_1 \mu_2 \dots \mu_n \left(\frac{\nu_1}{\mu_1} + \frac{\nu_2}{\mu_2} + \dots + \frac{\nu_n}{\mu_n} + n - 1 \right).$$

If the systems are pencils, then

$$\mu_i = 1 \quad , \quad \nu_i = 2\,(r_i - 1)\,;$$

thus *the order of the variety of contact* $RV_{12\,.\,n}$ *is*:

$$2\,(r_1 + r_2 + \ldots + r_n) - n - 1.$$

4. Returning to the correspondence between the points $Q_{12\ldots n}$ and Q_{n+1} we find for the number of coincidences which are points of intersection of l with the demanded locus L, i. e. for the order of L:

$$a_1 r_1 + a_2 r_2 + \ldots + a_{n+1} r_{n+1} - 2\,(r_1 + r_2 + \ldots + r_{n+1}) + \\ + n + 1 = \sum_{i=1}^{i=n+1} \{(a_i - 2)\,r_i + 1\}.$$

It is easy to see that a base-variety B_i of the pencil (V_i) is an $(a_i - 1)$-fold variety of L. The tangential spaces Sp^{n-1} of L in a point P of B_i are the tangential spaces in P of the varieties V_i, which are laid successively through one of the $a_i - 1$ points of intersection (not lying on P and the base-varieties) of the varieties $V_1, V_2, \ldots, V_{i-1}, V_{i+1}, \ldots, V_{n+1}$ passing through P.

So we find:

Given $n+1$ *pencils* $(V_i)\,(i = 1, 2, \ldots, n+1)$ *of* $(n-1)$*-dimensional varieties in the space of operation* Sp^n. *Let* r_i *be the order of the varieties of the pencil* (V_i) *and* a_i *the number of the points of intersection (not lying on the base-varieties) of arbitrary varieties of the pencils* $(V_1), (V_2), \ldots, (V_{i-1}), (V_{i+1}), \ldots, (V_{n+1})$. *The locus proper of the pairs of points lying on varieties of each of the pencils is an* $(n-1)$*-dimensional variety of order*

$$\sum_{i=1}^{i=n+1} \{ (a_i - 2)\, r_i + 1 \},$$

having the $(n-2)$*-dimensional base-variety of pencil* (V_i) *as* $(a_i - 1)$*-fold variety.*

If $n > 3$, then also in the general case the base-varieties of the different pencils will intersect each other. In like manner as we have dealt with pencils of surfaces [1]) we can also determine the multiplicity of common points, curves etc. of base-varieties.

Sneek, Jan. 1907.

[1]) See page 555.