Astromony. — *"On the astronomical refractions corresponding to a distribution of the temperature in the atmosphere derived from balloon ascents."* Preliminary paper by H. G. VAN DE SANDE BAKHUYZEN.

1. The various theories of the astronomical refraction in our atmosphere consider the atmosphere as composed of an infinite number of concentric spherical strata, each of uniform density, whose centre is the centre of the earth and whose densities or temperatures and refractive powers vary in a definite way.

The various relations between the temperature of the air and the height above the surface of the earth, assumed in the existing theories, are chosen so, that 1st they do not deviate too far from the suppositions on the distribution of the temperature in our atmosphere, made at the time when the theory was established, 2nd that the formula derived from this relation for the refraction in an infinitesimal thin layer at any altitude could be easily integrated.

At the time when the various theories were developed, only little was known about the variations of the temperature for increasing heights, and this little was derived from the results of a small number of balloon ascents and from the observations at a few mountain-stations. In the last decade, however, ascents of manned as well as of unmanned balloons with self-registering instruments have greatly increased in number, and our knowledge of the distribution of the atmospheric temperature has widened considerably, and has become much more accurate. Now I wish to investigate, whether by means of the data obtained, we can derive a better theory of refraction, or if it will be possible to correct the results of the existing theories.

2. The temperatures in our atmosphere at different heights have been derived from the following publications:

I. Ergebnisse der Arbeiten am aëronautischen Observatorium Tegel 1900—1902, Band I, II and III.

II. Travaux de la station Franco-scandinave de sondages aëriens à Halde par Teisserenc de Bord. 1902—1903.

III. Veröffentlichungen der internationalen Kommission für wissenschaftliche Luftschiffahrt.

From the last work I have only used the observations from December 1900 till the end of 1903.

I wished to investigate the distribution of the temperature up to the greatest heights, and therefore I used for my researches only the balloon ascents which reached at least an elevation of 5000 meters;

and, following HERGESELL's advice, I have used only the temperatures observed during the ascents, as during the descents aqueous vapour may condense on the instruments.

It is evident that for the determination of the refraction, as a correction to the results of the astronomical observations, we must know the variations of the temperature at different heights with a clear sky. For the temperatures, especially of the layers nearest to the surface of the earth, will not be the same with cloudy and uncloudy weather, as in the first case the radiation of the earth will lower the temperature of those layers, and so cause an abnormal distribution of temperature. It is even possible that in the lower strata the temperature rises with increasing height, instead of lowering, as is usual.

For this reason I have divided the balloon ascents into two groups, 1st those with a cloudy sky, 2nd those with a clear or a partly clouded sky.

In working out the observations, I have supposed that for each successive kilometer's height the temperature varies proportionally to the height, and after the example of meteorologists, I have determined the changes of temperature from kilometer to kilometer. For this purpose, I have selected from the observations, made during each ascent, the temperature-readings on those heights, which corresponded as nearly as possible with a round number of kilometers, and I have derived the variations of temperature per kilometer through division.

The available differences of height were often less than a kilometer, especially at the greatest elevations; in those cases I adopted for the weight of the gradient a number proportional to the difference of heights. Sometimes on the same day, at short, intervals several ascents have been made at the same station, or at neighbouring stations, from which the variations of temperature at the same heights could be deduced. In these cases I have used the mean of the results obtained, but I assumed for that mean result the same weight as for a single observation, as the deviations of the daily results from the normal distribution of temperature are only for a small part due to the instrumental errors, and for the greater part to meteorological influences.

3. The observations which I have used, were the following: from publication I, 31 ascents of which 12 had been made in pairs on the same day, so that 25 results were obtained; from publication II, 38 ascents all on different days; and from publication III, 170

ascents distributed over 119 different days; — I have disregarded the observations marked as uncertain in this work. On the whole I have obtained the results on 182 different days, of which 58 with unclouded and 124 with clouded sky.

The temperature gradients for each month were derived from this material, and to obtain a greater precision, I have combined them in four groups, each of three successive months, December, January and February (winter), March, April and May, (spring), June, July and August, (summer), September, October, November, (autumn).

TABLE I.

Variations of temperatures per kilometer.

(V.T. Variation of temperature per kilometer; N. Number of observations).

A. Clear sky.

	Winter.		Spring.		Summer.		Autumn		Mean.	
Kil	V.T.	N.	V.T.	N.	V.T.	N.	V.T.	N.	V.T.	N.
0— 1	+ 1°.2	10	— 3°.6	15	— 2°.8	18	+ 0°.6	15	— 0°.6	58
1— 2	— 4.2	10	— 5.4	15	— 4.3	18	— 3.2	15	— 4.3	58
2— 3	— 5.2	10	— 4.9	15	— 4.4	18	— 4.6	15	— 4.7	58
3— 4	— 5.4	10	— 5.8	15	— 5.4	18	— 5.3	15	— 5.5	58
4— 5	— 5.3	10	— 6.7	14.3	— 5.9	18	— 5.7	14.9	— 5.9	57.2
5— 6	— 5.6	8.9	— 7.1	13.6	— 6.0	18	— 7.3	13.8	— 6.5	54.3
6— 7	— 5.8	8	— 7.5	12.7	— 6.6	17.3	— 6.7	10.1	— 6.7	48.1
7— 8	— 6.8	7	— 7.8	10.8	— 7.5	14.6	— 8.0	8	— 7.5	40.4
8— 9	— 7.6	5	— 6.4	7.8	— 7.4	13.3	— 8.1	8	— 7.3	34.1
9—10	— 5.9	4	— 4.4	5.7	— 7.2	13	— 6.9	7	— 6.4	29.7
10—11	— 3.8	2.9	— 2.5	5	— 6.8	10.4	— 6.1	6.8	— 5.4	25.1
11—12	— 6.2	2	— 2.4	2.6	— 5.9	5.2	— 2.0	5.9	— 3.5	15.7
12—13	— 1.6	2	+ 2.0	1	— 1.1	2	— 1.0	4.9	— 0.7	9.9
13—14			+ 7.0	1	+ 1.0	2	— 4.0	1.6	— 0.8	4.6
14—15					+ 0.7	1.6	— 5.1	1	— 1.5	2.6
15—16					+ 0.8	1			+ 0.8	1

B. Cloudy sky.

	Winter.		Spring.		Summer.		Autumn.		Mean.	
Kil.	V.T.	N.	V.T.	N.	V.T.	N.	V.T.	N.	V.T.	N.
0— 1	— 1°.8	27	— 5°.5	33	— 3°.7	24	— 3°.9	40	— 3°.8	124
1— 2	— 3.0	27	— 5.6	32.5	— 5.1	24	— 3.7	49	— 4.3	123.5
2— 3	— 4.5	27	— 4.8	33	— 5.1	24	— 4.3	40	— 4.5	124
3— 3	— 5.8	27	— 5.5	33	— 5.1	23.8	— 5.8	39.5	— 5.6	123.3
4— 5	— 6.8	27	— 6.7	33	— 6.1	23	— 6.1	39	— 6.4	12.2
5— 6	— 6.9	26	— 6.7	30.7	— 6.7	21.5	— 6.2	36.5	— 6.6	114.7
6— 7	— 6.8	25.4	— 6.7	25	— 6.6	17.7	— 7.3	27.8	— 6.9	95.9
7— 8	— 6.9	19.7	— 7.2	20.3	— 7.2	16.8	— 5.9	21.6	— 6.8	78.4
8— 9	— 6.1	14.2	— 6.0	16.2	— 7.9	14.1	— 7.9	13	— 6.9	57.5
9—10	— 6.2	12.3	— 3.9	12.9	— 8.4	12.1	— 7.5	11.4	— 6.5	48.7
10—11	— 5.4	9.4	— 1.8	9.6	— 5.9	8.1	— 5.4	8.5	— 4.5	35.6
11—12	— 2.5	7.6	+ 1.0	8.3	— 2.1	5.1	— 1.9	6.8	— 1.2	27.8
12—13	— 1.3	5	+ 1.2	6.7	+ 0.2	1.9	— 0.5	4.1	+ 0.1	17.7
13—14	— 0.9	2.7	— 3.9	1			+ 1.7	1.4	— 0.8	5.1
14—15	+ 1.9	1.9	— 3.2	1					+ 0.2	2.9
15—16	— 0.6	1	— 3.2	0.5					— 1.5	1.5
16—17	+ 0.1	0.8							+ 0.1	0.8

We may derive from these tables that the mean variation of temperature with clear and with cloudy weather only differs in the lower strata, but is nearly the same in the higher ones.

In order to deduce from the numbers in this table the temperatures themselves from kilometer to kilometer, I have also derived from the data the following mean temperatures at the surface of the earth:

	clouded sky	clear sky
Winter	+ 0°.1	— 0°.9
Spring	+ 6 .4	+ 5 .1
Summer	+ 14 .4	+ 14 .7
Autumn	+ 9 .0	+ 7 .9

By means of these initial temperatures and the gradients of table I

C. Cloudy and uncloudy sky.

	Winter.			Spring.			Summer.			Autumn.			Mean.		
Kil.	V.T.	N.	Hann.	V.T.	N.	Hann.	V.T.	N.	Hann.	V.T.	N.	Hann.	V.T.	N.	Hann.
0— 1	−1°.0	37	+0°.5	−3°.8	48	−3°.1	−3°.3	42	−4°.1	−2°.7	55	−2°.1	−2°.8	182	−2°.2
1— 2	−3.3	37	−2.2	−5.6	47.5	−5.5	−4.7	42	−5.6	−3.6	55	−4.1	−4.3	181.5	−4.4
2— 3	−4.7	37	−5.0	−4.9	48	−5.1	−4.8	42	−5.1	−4.4	55	−4.8	−4.7	182	−5.0
3— 4	−5.7	37	−5.8	−5.6	48	−5.8	−5.2	41.8	−5.6	−5.6	54.5	−5.8	−5.5	181.3	−5.7
4— 5	−6.4	37	−6.7	−6.7	47.3	−6.7	−6.0	41	−5.9	−6.0	53.9	−5.9	−6.2	179.2	−6.3
5— 6	−6.7	34.9	−6.7	−6.8	44.3	−7.3	−6.4	39.5	6.4	−6.5	50.3	−6.8	−6.6	169	−6.8
6— 7	−6.6	33.4	−6.7	−7.0	37.7	−7.2	−6.6	35	−7.2	−7.1	37.9	−7.1	−6.9	144	−7.0
7— 8	−6.9	26.7	−7.2	−7.4	31.1	−6.3	−7.4	31.4	−7.7	−6.4	29.6	−7.3	−7.3	118.8	−7.1
8— 9	−6.5	19.2	−6.9	−6.1	24	−6.4	−7.6	27.4	−7.6	−7.9	21	−7.6	7.1	91.6	−7.1
9—10	−6.2	16.3	−6.4	−4.0	18.6	−4.8	−7.8	25.1	−6.9	−7.4	18.4	−6.6	−6.5	78.4	−6.3
10—11	−5.0	12.3	−3.9	−2.0	14.6	−0.9	−6.4	18.5	−5.0	−5.7	15.3	−6.1	−4.9	60.7	−4.0
11—12	−2.6	9.6	0.0	−0.2	10.	+0.5	−4.0	10.3	−2.4	−1.9	12.7	−2.7	−2.1	43.5	−1.2
12—13	−1.2	7		+1.3	7.7		−0.5	3.9		−0.5	9		−0.2	27.6	
13—14	−0.9	2.7		−1.6	2		+1.0	2		−1.3	3		−0.8	9.7	
14—15	+1.9	1.9		−3.2	1		+0.7	1.6		−5.1	1		−0.6	5.5	
15—16	−0.6	1		−3.2	0.5		+0.8	1					−0.6	2.5	
16—17	+0.1	0.8											+0.1	0.8	

which in a few cases have been slightly altered, I have derived the following list of temperatures for clear weather from kilometer to kilometer.

Although the adopted values for the temperature of the air above 13 kilometer are not very certain, yet the observations indicate that at these heights the temperature decreases slowly with increasing height. The refraction in those higher strata being only a small part of the computed refraction, nearly $^1/_{10}$, an error in the adopted distribution of temperature will have only a slight influence on my results.

I must remark that almost all the observations have been made during the day, generally in the morning. It is evident that the variation of temperature, especially near the surface of the earth, is not the same during the day and during the night, but the number of

TABLE II.

Temperatures at heights from 0 to 16 kilometer for clear weather.

	Winter.		Spring.		Summer.		Autumn.		Mean.	
Height.	Temp.	Diff.	Temp.	Diff.	Temp.	Diff.	Temp.	Diff.	Temp.	Diff.
0	− 1°.9		+ 5°.1		+14°.7		+ 7°.9		+ 6°.4	
		+1.2		−3.6		−2.8		+0.6		−1.1
1	− 0.7		+ 1.5		+11.9		+ 8.5		+ 5.3	
		−4.2		−5.4		−4.3		−3.2		−4.3
2	− 4.9		− 3.9		+ 7.6		+ 5.3		+ 1.0	
		−5.2		−4.9		−4.4		−4.6		−4.8
3	−10.1		− 8.8		+ 3.2		+ 0.7		− 3.8	
		−5.4		−5.8		−5.4		−5.6		−5.5
4	−15.5		−14.6		− 2.2		− 4.9		− 9.3	
		−5.8		−6.7		−5.9		−6.1		−6.1
5	−21.3		−21.3		− 8.4		−11.0		−15.4	
		−6.0		−6.7		−6.0		−6.9		−6.4
6	−27.3		−28.0		−14.1		−17.9		−21.8	
		−6.2		−6.9		−6.6		−7.2		−6.7
7	−33.5		−34.9		−20.7		−25.1		−28.5	
		−6.8		−7.3		−7.3		−7.7		−7.3
8	−40.3		−42.2		−28.0		−32.8		−35.8	
		−7.3		−6.9		−7.6		−7.6		−7.4
9	−47.6		−49.1		−35.6		−40.4		−43.2	
		−6.4		−5.4		−7.2		−6.9		−6.4
10	−54.0		−54.5		−42.8		−47.3		−49.6	
		−4.9		−2.5		−6.8		−6.1		−5.1
11	−58.9		−57.0		−49.6		−53.4		−54.7	
		−2.1		−1.0		−4.0		−2.0		−2.3
12	−61.0		−58.0		−53.6		−55.4		−57.0	
		−1.0		−1.0		−1.0		−1.0		−1.0
13	−62.0		−59.0		−54.6		−56.4		−58.0	
		−0.6		−0.6		−0.6		−0.6		−0.6
14	−62.6		−59.6		−55.2		−57.0		−58.6	
		−0.4		−0.4		−0.4		−0.4		−0.4
15	−63.0		−60.0		−55.6		−57.4		−59.0	
		−0.2		−0.2		−0.2		−0.2		−0.2
16	−63.2		−60.2		−55.8		−57.6		−59.2	

observations was not great enough for a *reliable* determination of this difference. Lastly I remark that the various balloon ascents have been made from different stations, Halde (in Danemark), Berlin, Paris, Strasbourg and Vienna and consequently the given values do not hold for one definite place, but for the mean of the *area* enclosed by those stations.

After I had derived the temperatures given in table II, I got notice of two papers, treating of about the same subject, namely: J. HANN, Ueber die Temperaturabnahme mit der Höhe bis zu 10 Km. nach den Ergebnissen der internationalen Ballonaufstiege. Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der K. K. Akademie der Wissenschaften Wien. Band 93, Abth. IIa, S. 571; and S.

Grenander. Les gradients verticaux de la température dans les minima et les maxima barométriques. Arkiv för Matematik, Astronomi och Fysik. Band 2. Hefte 1—2 Upsala, Stockholm.

Of the results which Hann has given, up to a height of 12 kil., I have taken the means of groups of 3 months, which are printed in table I by the side of the values I had obtained; the agreement of the two results, which for the greater part have been deduced from different observations, is very satisfactory.

Grenander in his paper chiefly considers the relation between the changes of temperature and the barometer readings: his results cannot therefore be compared with mine directly, but probably we are most justified in comparing the variations of temperature at barometer maxima, with those which I have computed for clear weather. For great elevations, till nearly 16 kil., Grenander also obtains with increasing height a small decrease of temperature.

It is difficult to state with what degree of precision the temperatures of table II represent the mean values for the different seasons; the deviations, especially at great heights, may perhaps amount to some degrees, but certainly they represent the mean distribution of temperature better than the values adopted in the various theories of refraction, and we can therefore derive from them more accurate values for the refraction.

4. It is hardly possible to represent the relation between the temperatures in table II and the heights by a simple formula, and to form a differential equation between the refraction, the zenith distance and the density of the atmosphere at a given height, which can be easily integrated.

Therefore I have followed another method to determine the refraction corresponding to the distribution of temperature I had assumed.

According to Radau's notations (Essai sur les réfractions astronomiques. Annales de l'Observatoire de Paris. Mémoires Tome XIX), the differential equation of the refraction, neglecting small quantities, is:

$$ds = \alpha'' \frac{\left(1 - \frac{l_0}{R}(\eta - 3\,\varepsilon\omega)\right) d\omega}{\sqrt{\left\{cot^2 z + 2\frac{l_0}{R}(\eta - \varepsilon\omega) - \left(\frac{l_0}{R}\eta - 2\frac{l_0}{R}\varepsilon\omega\right)^2\right\}}} \;. \qquad (I)$$

Here is:

R radius of the earth for 45° latitude,

r_0 radius of the earth for a given point,

h height above the surface of the earth,

$r = r_0 + h$,
μ_0 index of refraction at the surface of the earth,
μ „ „ „ „ „ height h,
ϱ_0 density of the air at the surface of the earth,
ϱ density at the height h,
t_0 temperature at the surface of the earth,
l_0 height of a column of air of uniform density at 45° latitude, of a temperature t_0, which will be in equilibrium with the pressure of one atmosphere, the gravity being the same at different heights. According to REGNAULT's constants, we have $l_0 = 7993\ (1 + at_0)$ meter, if a represents the coefficient of expansion of the air.

Between these quantities exist the following relations:

$$\mu^2 = 1 + 2\,c\varrho \quad (c \text{ being a constant}), \quad \omega = 1 - \frac{\varrho}{\varrho_0}$$

$$\alpha = \frac{c\varrho_0}{1 + 2c\varrho_0} \qquad \alpha'' = \frac{\alpha}{\sin 1''} \qquad \varepsilon = \frac{R}{l_0}\,\alpha \qquad y = \frac{Rh}{(r_0 + h)l_0}.$$

To determine the value of ds at each height, we require a relation between ω and y or between ω and h, which can be obtained when we assume that the temperature varies according to IVORY's theory, or that the temperature varies as represented in table II. For the same given values of z and ω, the two values of ds in formula (1) can be computed by means of the first and by means of the second supposition, and the differences of these two values of ds can be found. By means of mechanical quadrature, we can then determine the differences Δs of the refractions s according to IVORY's theory and according to table II.

The relations between y and ω may be found in the following manner.

5. If in a given horizontal initial plane, at a distance r_0 from the centre of the earth, the pressure is p_0, the temperature t_0 and the density of the air ϱ_0, and in another horizontal plane, h kil. above the former, the pressure is p, the temperature t, the distance from the centre of the earth r, and the density of the air ϱ, then we have (see RADAU):

$$l_0\, d\left(\frac{p}{p_0}\right) = -\frac{\varrho}{\varrho_0}\left(\frac{R}{r}\right)^2 dh = -\frac{\varrho}{\varrho_0}\frac{R}{r_0}\, d\left(\frac{Rh}{r}\right), \text{ or } \frac{r_0}{R}\, d\left(\frac{p}{p_0}\right) = -\frac{\varrho}{\varrho_0}\, d\left(\frac{Rh}{l_0\, r}\right),$$

or, putting $\dfrac{\varrho}{\varrho_0} = \eta$ and $\dfrac{Rh}{(r_0 + h)\, l_0} = y$:

$$\frac{r_0}{R}\, d\left(\frac{p}{p_0}\right) = -\eta\, dy; \qquad \text{(II)}$$

further is :

$$\frac{p}{p_0} = \frac{1+at}{1+at_0}\,\eta = \left(1 - \frac{a\,(t_0-t)}{1+at_0}\right)\eta = (1-\vartheta)\,\eta, \quad . \quad . \quad \text{(III)}$$

if we put $\dfrac{a\,(t_0-t)}{1+at_0} = \vartheta$.

When dividing equation (II) by (III), we get:

$$\frac{r_0}{R}\,\frac{d\left(\dfrac{p}{p_0}\right)}{\dfrac{p}{p_0}} = -\frac{dy}{1-\vartheta},$$

while by differentiating logarithmically we find:

$$\frac{d\left(\dfrac{p}{p_0}\right)}{\dfrac{p}{p_0}} = -\frac{d\vartheta}{1-\vartheta} + \frac{d\eta}{\eta}.$$

From the two last equations follows:

$$dy = \frac{r_0}{R}\left\{d\vartheta - (1-\vartheta)\frac{d\eta}{\eta}\right\} = \frac{r_0}{R}\left\{d\vartheta + (1-\vartheta)\frac{d\omega}{1-\omega}\right\}. \quad . \quad \text{(IV)}$$

According to IVORY's theory $\vartheta = f\omega$, where f is a constant value (RADAU assumes 0,2); if we introduce this relation into the equation (III) we obtain after integration:

$$y = 0{,}4\,\frac{r_0}{R}\,\omega - 1{,}8420681\,\frac{r_0}{R}\,Br.\,log\,(1-\omega) \quad . \quad . \quad . \quad \text{(V)}$$

By substituting (V) in (I) we can therefore calculate for each value of ω the value of ds according to IVORY's theory.

6. Now I proceed to determine the relation between ω and y according to the temperature table II.

Of two horizontal planes, one above the other, the first is situated n kil. (n a whole number), the second n' kil. ($n' =$ or $< n+1$) above the surface of the earth; their distances from the centre of the earth are r_n and $r_{n'}$, their temperatures t_n and $t_{n'}$ and the values of y, y_n and $y_{n'}$. The temperature between n and n' varies regularly with the height and, to simplify the formulae, I suppose $t_n - t_{n'}$ proportional to $y_{n'} - y_n$, so that, if $\vartheta_n = \dfrac{a\,(t_n - t_{n'})}{1+at_n}$:

$$\frac{R}{r_n}\,(y_{n'} - y_n) = c_n\,\vartheta_n\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VI)}$$

Hence follows $\dfrac{R}{r_n}\,dy = c_n\,d\vartheta$ and after substitution of dy in (IV) and integration

$$(c_n - 1)\, lg\,(1 - \vartheta_n) = lg\,(1 - \omega) \quad . \quad . \quad . \quad . \quad (\mathrm{VII})$$

in which $1 - \omega$ represents the ratio of the densities in the two horizontal planes.

If we substitute $n + 1$ for n', we can find in table II the temperature for the two planes and hence also ϑ_n; as y_n and y_{n+1} are also known, we can derive from (VI) the value of c_n and we can deduce from (VII) the ratio of the densities in those planes. By putting for n successively 0, 1, 2, etc. we can construct a table containing the densities of the air, D_1, D_2, D_3 etc. at the height of 1, 2, 3, etc. kil. above the surface of the earth, the density at the surface being unity.

It is easy to derive from this table the height of a layer of a given density d. If $d < D_n$ and $> D_{n+1}$, the layer must be situated between n and $n + 1$ kil., and we only want to know in which manner, within this kil., the density varies with the height h above the lower plane.

We may assume:

$$\frac{d}{D_n} = 10^{-ah}.$$

For $h = 1$ kil., $d = D_{n+1}$, hence $a = -\, lg\, \dfrac{D_{n+1}}{D_n}$.

a being known, we may determine for each value of d, h and also y. By substitution in (I) we find then for each value of ω the value of ds.

7. Now we are able to form the differences of ds after the theory of Ivory and after the table of temperatures II, for values of ω which increase with equal amounts, and then determine the whole difference of the refraction for both cases.

For great values of z and small values of y and ω the coefficients of $d\omega$ in (I) will become rather large, which derogates from the precision of the results.

This will also be the case when the differences of the successive values of ω are large; small differences are therefore to be preferred, but they render the computation longer.

Both these difficulties can be partly avoided if, according to Radau's remark, we introduce $\sqrt{\omega}$ as a variable quantity instead of ω: the value of ds thus becomes:

$$ds = \frac{a''}{\sqrt{\frac{l_0}{2R}}} \frac{\left(1 - \frac{l_0}{R}(y - 3\,\varepsilon\omega)\right) d\sqrt{\omega}}{\sqrt{\left\{\frac{\frac{R}{2l_0}\cot^2 z + y}{\omega} - \varepsilon - \frac{l_0}{2R\omega}(y - 2\varepsilon\omega)^2\right\}}} \qquad \text{(VIII)}$$

or approximately:

$$ds = \frac{a''}{\sqrt{\frac{l_0}{2R}}} \frac{d\sqrt{\omega}}{\sqrt{\left\{\frac{\frac{R}{2l_0}\cot^2 z + y}{\omega} - \varepsilon\right\}}}$$

It is evident that for small values of ω the coefficient of $d\sqrt{\omega}$ in (VIII) is smaller than that of $d\omega$ in (I), and that also the refraction in the lower strata will be found more accurately by means of the formula (VIII) than by means of (I). For if we increase $\sqrt{\omega}$ in formula (VIII) and ω in formula (I) with equal quantities, beginning with zero, we find, that, from $\omega = 0$ to $\omega = 0{,}2$, the number of values in the first case is twice as great as in the second case, hence the integration by means of quadrature will give more accurate results in the first case.

Therefore I have used the formula (VIII) and computed the coefficient of $d\sqrt{\omega}$ for values of $\sqrt{\omega}$, 0, 0,05, 0,10, 0,15 . . . to 0,95.

The density of the air which corresponds to $\sqrt{\omega} = 0{,}95$, occurs at the height of about 18 kil. From the observations at my disposal I could not deduce reliable values for the temperature at heights above 16 kil.; yet it is probable that the gradients at those heights are small and I have assumed the temperature at heights of 17 and 18 kil. to be equal to that at a height of 16 kilometers.

In this way I have determined by means of mechanic quadrature, and an approximate computation of the refraction between $\sqrt{\omega} = 0.925$ and $\sqrt{\omega} = 0.95$, the differences Δs of the two values of the refraction corresponding to IVORY's theory and corresponding to the table of temperature II in the part of the atmosphere between the earth's surface and a layer at a height of about 18 kil. where $\sqrt{\omega}$ is 0.95. I have worked out this comptuation for the zenith distances 85°, 86°, 87°, 88°, 88°30′, 89°, 89°20′, 89°40′ and 90°.

An investigation, made for the purpose, showed me that in formula (VIII) the terms $\frac{lo}{R}(y - 3\varepsilon\omega)$ in the numerator and $\frac{lo}{2R\omega}(y - 2\varepsilon\omega)^2$ in the denominator may be neglected for all zenith distances except $z = 90°$; therefore I have taken them into account in the computation of the horizontal refraction only.

The results which I have obtained for the differences:

$\Delta s =$ IVORY—table of temperatures

are the following:

TABLE III.

Refraction after IVORY — Refractions after the table of temperatures II.

Zenith distance	Winter	Spring	Summer	Autumn	Annual mean	Annual mean — Winter	Annual mean — Spring	Annual mean — Summer	Annual mean — Autumn
85°	+ 0″21	+ 0″78	+ 0″66	+ 0″31	+ 0″49	+ 0″28	— 0″29	— 0″17	+ 0″18
86°	+ 0.43	+ 1.26	+ 0.95	+ 0.30	+ 0.66	+ 0 53	— 0.60	— 0 29	+ 0.36
87°	– 0.47	+ 2.08	+ 1.31	— 0.20	+ 0 66	+ 1 13	— 1.42	— 0.65	+ 0.86
88°	— 3 93	+ 3.40	+ 0.95	— 3.29	— 0.83	+ 3 40	— 3.93	— 1.78	+ 2.46
88°30′	— 9.64	+ 3.06	— 0.67	— 8.51	— 3.95	+ 5.69	— 7.01	— 3.28	+ 4.56
89°	—23.69	+ 1.08	— 5.45	—21.45	—12.31	+11.38	—13.39	— 6 86	+ 8.84
89°20′	—43.80	— 3.17	—12 68	—38.77	—24 51	+19.29	—21 34	—11.83	+14.26
89°40′	—1′ 24″97	—13.07	—25.25	—1′ 14″16	—27.74	+34.23	—34.67	—22.49	+23.42
90°	—2 32.4	—33.1	—52.9	—2 9.6	—1′30″9	+1′ 1″5	—57.8	—38.0	+38.7

To test the computations, we may compare the mean of the values of Δs for the four seasons, and the values of Δs in column 6 which have been computed, independently the former, for the mean yearly temperatures, which are almost equal to the mean of the temperatures in the four seasons. Only for $z = 89°40'$ and $z = 90°$ do these values show deviations exceeding 0″.1.

From table III follows, 1 that for a distribution of temperature, as derived by me from observations, the refraction deviates perceptibly from that deduced from IVORY's theory, 2 that the differences in the refraction in the different seasons are about of the same order as the deviations themselves. I want it to be distinctly understood, 1 that the adopted distribution of temperature above 13 kil. and especially from 16 to 18 kil. is uncertain, and 2 that I have not taken into account the refraction in the layers which are lying more than 18 kil. above the surface of the earth, in other words those layers where the density, as compared to that of the surface of the earth, is less than $1 - 0{,}95^2$, or less than 0,0975.