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Meteorology. — *"The treatment of wind-observations."* By Dr. J. P. VAN DER STOK.

(Communicated in the meeting of February 23, 1907).

1. When working out wind-observations we directly meet with the difficulty that a method holding generally, in which the characteristics of a wind distribution come to the fore in condensed form, does not exist. The discussion held for many a year concerning the desirability or not of an application of LAMBERT's formula, i. e. of the calculation of the vectorial mean of velocity or force has not led to a definite result and the consequence is that for regions where trade- and monsoon winds prevail the calculation of this mean can be applied, not for higher latitudes, so that here we have to judge by extensive tables of frequencies of direction and mean velocities, independent of direction.

When working out the wind-observations made at Batavia I did not hesitate to make an extensive use of this formula; the same method has been followed in the atlas for the East Indian Archipelago; but in order to give at least a notion of the value of the velocities annulling each other here I have added to the resulting movement (called by HANN *windpath*) a so-called factor of stability. If namely the wind were perfectly stable, the vectorial mean would be equal to the mean independent of the direction and the stability would amount to 100 %, which percentage becomes smaller and smaller according to the direction of the wind becoming more variable. So here attention is drawn to the fact, that a part of the observations is eliminated, but it is not indicated what character this vanishing part has which becomes chief in our regions.

In the climatological atlas lately published of British India the same method is followed; in the "Klima Tabeller for Norge" MOHN gives but the above mentioned tables without calculation of the vectorial mean, which is, indeed, of slight importance for this climate.

The same uncertainty is found in the graphical representation of a wind distribution by so-called windroses; almost everyone who has been occupied in arranging books of prints has projected windroses of his own; some of those roses, as e. g. in the "Vierteljahrskarte für die Nordsee und Ostsee" published by the "Deutsche Seewarte", show only the frequencies of direction without velocities; in others, as e. g. those shown in the above atlas of the East Indies, each direction is taken into account with the velocity belonging to it as weight, so that mean velocities are represented. All these roses furnish discontinuous quantities and change their aspect according to their boundaries being taken differently.

In BUCHAN's general meteorological atlas no roses are projected, only arrows indicating the most frequent direction without heeding the force, and in the "Segelhandbuch für den Atlantischen Ozean" published by the "Deutsche Seewarte" for higher latitudes where the wind is variable the use of wind-observations is entirely done away with and arrows have been drawn in accordance with the course of the mean isobars on account of the law of BUYS BALLOT, where a constant angle of 68° between gradient and direction of wind has been assumed.

This short survey of the manner in which in the most recent standard works this problem has been treated may show that indeed there is as yet no question about a satisfactory solution, as has already been observed.

The aim of this communication is to hit upon a general method of operation and representation of an arbitrary wind distribution in which to the variable part also justice is done, whilst the graphical representation has a continuous course and shows at a glance the five characteristic quantities which mark each wind distribution and which may be, therefore, called the *wind-constants*.

The method proposed here is founded on the basis of the calculus of probability, but it is important to notice that it is not at all bound to it; at the bottom it is the same which is generally applied in the treatment of directed quantities: distribution of masses and forces in mechanics, the theory of elasticity, the law of radiation and the theory of errors in a plane.

2. A wind-observation can be represented by a point in a plane such that the distance to an assumed origin is a measure for the velocity of the wind (or force) and that the angle made by the radius vector with the Y (North) axis counted from N. to E. indicates the direction. If in this way all observations, N in number, are drawn and if we think that to each point an equal mass is connected, then in general the centre of gravity will not coincide with the origin selected; its situation may be determined by the quantities R_0 and α. The distribution of the masses around the centre of gravity, is then characterized by the lengths M and M' of the two principal axes of inertia and the angle β enclosed by the axes M and Y.

As is known the five constants by which such a system is characterized can be calculated according to this purely mechanic notion by determining the moments M_x and M_y with respect to the axes and the moments of inertia M_x^2 and M_y^2 and M_{xy}, which furnish the five equations necessary for the calculation of the unknown quantities.

We arrive at quite the same equations when the distribution of the winds according to direction and velocity is regarded as a system of accidental, directed quantities in a plane. The centre of gravity then represents according to size and direction the constant part of the wind which is supposed to be connected with all observations and of which, therefore, the probability is equal to unity; the axes of inertia become principal axes of probability and the lengths M and M' are replaced by the reciprocal lengths h and h', so that

$$h^2 = \frac{1}{2M^2} \quad , \quad h'^2 = \frac{1}{2M'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The sum of the masses is put equal to unity and for the probability that an observation lies between the limits R and $R + dR$ of velocity and θ and $\theta + d\theta$ as far as direction is concerned the expression holds

$$\frac{hh'}{\pi} e^{-f(R,\theta)} R\, dR\, d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where:

$$f(R,\theta) = h^2 [R \cos(\theta - \beta) - R_0 \cos(\alpha - \beta)]^2 + h'^2 [R \sin(\theta - \beta) - R_0 \sin(\alpha - \beta)]^2 . (3)$$

In the language of the theory of errors (R_0, α) would be the so-called constant error, M and M' the greatest and smallest projections of the mean errors. As observations of wind agree still less than other meteorological quantities with the opinion held in the theory of errors, where the constant part is regarded as the end of the operation and the variable quantities as deviations, it is desirable when applying the calculus of probability to quantities of this kind to be entirely free of the terminology used in the theory of errors, but which would be here without meaning and which would give rise to misunderstanding.

The treatment must also differ somewhat from that of erroneous quantities, it being if not impossible at least impractical to correct all the observations for the constant part.

3. As examples of treatment two series of observations have been selected from the treated material.

a. Observations of wind performed at Bergen (Norway) during 20 years, 1885—1904, three times daily at 8 A.M, 2 P.M. and 8 P.M. The velocity (or force) of the wind is expressed in the so-called half scale of BEAUFORT (1—6) (Jahrbuch des Norwegischen Meteorol. Instituts, Christiania).

b. Observations of wind performed at Falmouth (Channel) during 17 years, 1874—1886 and 1900—1903; the observations made in the years 1887—1899 are published in such a way as to be useless for this investigation.

Observations have been used, made daily six times: at noon, 4 P.M., 8 P.M., midnight, 4 A.M. and 8 A.M.; the velocity of wind is expressed in English (statute) miles an hour (Hourly readings obtained from the selfrecording instruments etc. London).

With respect to the force of the wind estimated at Bergen is to be noticed that in this communication these scale-values are regarded not as forces but as velocities, although in reality they are neither one nor the other. According to a recent extensive investigation [1]) the ratio of the Beaufort values to corresponding velocities can be indicated by the following numbers

BEAUFORT	velocity meters a second	ratio	BEAUFORT	velocity meters a second	ratio
0	1.34	—	6	10.95	1.83
1	2.24	2.24	7	13.41	1.92
2	3.58	1.79	8	16.09	2.01
3	4.92	1.64	9	19.67	2.19
4	6.71	1.68	10	23.69	2.34
5	8.72	1.74			

As the various velocities do not appear in an equal number the total mean out of these ratios would not give a fit factor of reduction for mean BEAUFORT-values; so a certain weight must be assigned to each separate ratio. For this the frequencies have been used of the 36000 wind-velocities observed at Falmouth calculated for a whole year; in this way has been found for the reduction-factor 1.83; the English measure, miles an hour, can be reduced to m. a. s. and BEAUFORT scale-values by means of multiplication respectively by

0.447 and 0.244.

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1) The BEAUFORT scale of windforce.
Report of the Director of the Meteor. Office upon an Inquiry into the Relation between the Estimates of Windforce according to Admiral BEAUFORT's Scale and the velocities recorded by Anemometers. London, 1906.

4. The calculation of the five charateristic constants of a wind distribution amounts in one respect to the integration of (2), in another respect to the means applied in this integration to a given set of observations.

The integration of (2) takes place by the introduction of rectangular coordinates :

$$x = R \sin\theta \quad , \quad y = R \cos\theta,$$

where the element $R dR d\theta$ is replaced by the element $dx dy$, whilst the limits which were ∞ and 0 for R, 2π and 0 for θ, now become ∞ and $-\infty$.

Then the expression (2) under the sign of the integral is multiplied successively by

$$x,\ y,\ x^2,\ y^2 \text{ and } xy.$$

If we then put:

$$R_0 \cos(\alpha-\beta) = a, \qquad x = x' \sin\beta + y' \cos\beta,$$
$$R_0 \sin(\alpha-\beta) = b, \qquad y = x' \cos\beta - y' \sin\beta,$$

the variables x' and y' can be separated and the integration can be done; in this way we find for the determination of the five quantities to be obtained the five equations :

$$\left.\begin{aligned} M_x &= a\cos\beta - b\sin\beta, \quad M_y = a\sin\beta + b\cos\beta \\ M_{x^2} &= \frac{\cos^2\beta}{2h^2} + \frac{\sin^2\beta}{2h'^2} + a^2\cos^2\beta + b^2\sin^2\beta - ab\sin 2\beta \\ M_{y^2} &= \frac{\sin^2\beta}{2h^2} + \frac{\cos^2\beta}{2h'^2} + a^2\sin^2\beta + b^2\cos^2\beta + ab\sin 2\beta \\ 2M_{xy} &= \left(\frac{1}{2h^2} - \frac{1}{2h'^2}\right)\sin 2\beta + (a^2 - b^2)\sin 2\beta + ab\cos 2\beta \end{aligned}\right\} , \qquad (3)$$

out of which, on account of (1)

$$\left.\begin{aligned} M_x &= R_0\cos\alpha, \quad M_y = R_0 \sin\alpha \\ M_{x^2} + M_{y^2} &- [(M_x)^2 + (M_y)^2] = M^2 + M'^2 \\ M_{x^2} - M_{y^2} &- [(M_x)^2 - (M_y)^2] = (M^2 - M'^2)\cos 2\beta \\ 2M_{xy} &- 2M_x M_y = (M^2 - M'^2)\sin 2\beta \end{aligned}\right\} \quad . \ . \quad (4)$$

TABLE I. Frequencies of the wind.
Bergen. June.
In half BEAUFORT scale-values.

	0	1	2	3	4	5	Sum
C	261	—	—	—	—	—	261
N		59	30	29	5	2	125
NNE		6	6	1	—	—	13
NE		6	2	—	—	—	8
ENE		3	2	—	—	—	5
E		13	—	—	—	—	13
ESE		4	1	—	—	—	5
SE		24	3	1	—	—	28
SSE		40	16	3	—	—	59
S		115	54	22	6	—	197
SSW		56	39	15	4	—	114
SW		25	10	2	2	—	39
WSW		9	2	—	—	—	11
W		98	26	5	—	—	129
WNW		99	24	1	—	—	124
NW		190	51	6	—	—	247
NNW		246	118	46	12	—	422
Sum	261	993	384	131	29	2	1800

5. In order to apply the formulae (4) to a given set of observations we must compose for each period, e.g. each month, in the first place a table of frequency of direction and velocity, which can be easily done. In Table I such a composition has been given as an example.

Further out of this table have been calculated the products of these frequencies f with the scale-values R, the latter counted double, so that the products have been expressed in the ordinary BEAUFORT scale; finally these products have been once again multiplied by the corresponding scale-values (fR^2); in this simple way we find the sums.

TABLE II.

	fR	fR^2		fR	fR^2
N	472	2280	S	626	1500
NNE	42	156	SSW	390	1644
NE	20	56	SW	118	460
ENE	14	44	WSW	26	68
E	26	52	W	330	988
ESE	12	32	WNW	300	816
SE	66	180	NW	620	1792
SSE	162	524	NNW	1336	5296
			Som	4560	16888

The sums fR, multiplied respectively by $\cos\theta$ and $\sin\theta$ and divided by 1800, immediately furnish the quantities M_x and M_y; the sums fR^2 must be multiplied successively by $\cos^2\theta$, $\sin^2\theta$ and $\sin\theta\cos\theta$.

It is easier to multiply the latter sums by $\cos 2\theta$ and $\sin 2\theta$; if the total mean is S, we find:

$$M_x^2 = fR^2 \cos^2\theta = \tfrac{1}{2} S + \tfrac{1}{2} fR^2 \cos 2\theta$$
$$M_y^2 = fR^2 \sin^2\theta = \tfrac{1}{2} S - \tfrac{1}{2} fR^2 \cos 2\theta$$
$$2M_{xy} = fR^2 \sin 2\theta .$$

So the whole operation greatly resembles the calculation of FOURIER terms, indeed, also by the way of operation indicated here an analysis of the movement of the air is obtained.

In the Tables III and IV we find the values of the wind-constants calculated in this way; besides the five characteristic quantities we find still given as quantities practically serviceable for various ends:

$e = \dfrac{\sqrt{M^2 - M'^2}}{M}$, the excentricity of the ellipse of which M and M' represent the half principal axes,

(R_0' and α') the resultants of the squares of the velocities giving an image of the mean flux of energy,

$\overline{V}$ the mean velocity independent of the direction,

$\overline{V}^2$ the mean square of the velocity independent of the direction, i. e. a measure for the total energy; this quantity is according to (4) analogous to the square of the mean error, not corrected for the constant part, in the theory of errors,

N the number of used observations.

TABLE IIIa Constants of the wind.
Bergen 1885—1904.
In BEAUFORT scale-values.

	R_0	α	M	M'	β	e
January	1 84	179°	3 39	1.65	174°	0.873
February	1 51	180	3 10	1.59	172	0.858
March	1 16	183	3.25	1 60	171	0 872
April	0.40	2[illegible]5	2 88	1 47	169	0.859
May	0 68	284	2 84	1 37	169	0 879
June	1.07	302	2 61	1.19	171	0.891
July	0 93	276	2 64	1 22	168	0 885
August	0 73	251	2 86	1 22	168	0.904
September	0 97	212	2 88	1 38	174	0 876
October	1 10	182	2 94	1 52	171	0 857
November	1 51	179	3 10	1 48	171	0.880
December	1 78	179	3 14	1 57	174	0 866
Year	0 85	203	2 97	1 44	171	0.875

TABLE IIIb Constants of the wind.
Bergen 1885—1904.
In BEAUFORT scale-values.

	R'_0	α'	$\overline{V}$	$\overline{V^2}$	N
January	10.56	182°	3.31	17.63	1860
February	8 09	184	2 90	14.44	1695
March	5 96	186	3 06	14 48	1860
April	1 62	224	2 64	10 61	1800
May	2 62	281	2 59	10 39	1860
June	3 78	305	2.53	9.38	1800
July	3 19	271	2 58	9.32	1860
August	2 79	243	2 63	10.18	1860
September	4 30	215	2 67	11 14	1800
October	5 13	185	2 76	12.19	1860
November	7 77	182	2 91	14.09	1800
December	9.30	183	3 06	15.50	1860
Year	4 42	200	2 80	12 45	21915

TABLE IVa. Constants of the wind.
Falmouth. 1874—1886, 1900—1903.
In Eng. miles an hour.

	R_0	α	M	M'	β	e
January	5.23	211°	15.20	13.93	73°	0.400
February	5.33	209	14.08	13.25	164	0.339
March	3.56	255	15 02	13.26	67	0.470
April	0.85	215	13.70	12.21	72	0.454
May	2.34	255	12 02	11.52	40	0.286
June	4.60	255	11.19	10.07	158	0.436
July	6.46	254	10.39	8.92	155	0.507
August	5.79	245	10.48	9 76	82	0.363
September	3.77	229	11.05	10.67	164	0.260
October	3.56	239	13.51	13.03	81	0.266
November	4.57	248	13 75	13.03	93	0.318
December	5.69	243	13.69	12.98	22	0.318
Year	4 14	239	12.60	12.43	96	0.166

TABLE IVb. Constants of the wind.
Falmouth.

	R'_0	α'	$\overline{V}$	$\overline{V^2}$	N
January	146.0	205°	18 4	452.5	2821
February	159.9	203	16 9	402.1	2675
March	88.8	241	17.5	414.2	2930
April	23.0	178	15.7	337.4	2879
May	54.1	241	14.4	282.5	3140
June	92.4	252	13.7	247.8	3015
July	122.2	252	12.9	228.5	3060
August	122.0	242	13.0	238.7	3154
September	83.7	224	13.1	250.3	3047
October	78.5	223	16.4	370.0	3154
November	114.2	237	16.6	379.7	3053
December	148.6	233	16.6	388.4	2888
Year	98.2	229	15.4	330.4	35816

A closer discussion of the results arrived at in this way may for shortness' sake be left out; however, the observation is not superfluous that the two examples represent two types, a reason why they were chosen. At Bergen the ellipse of the variable winds is very constant of shape and the excentricity is very great; at Falmouth the difference between M and M' is always very slight and the differences found there are evidently to be regarded rather as accidental arithmetical results than as facts, the angle β being subject to great and irregular oscillations; evidently the ellipse approaches a circle, so that in form (2) we may put $h = h'$. This leading to a considerable simplification of the formula, these observations at Falmouth are eminently fit for comparison of the results of calculation and observation, whilst also the fact that here real velocities have been observed with well-verified instruments, makes this series very favourable.

6. The expression (2) shows: the probability that an observation lies between the limits R and $R + dR$, θ and $\theta + d\theta$; the same expression without the element $RdRd\theta$ indicates: the specific probability of a wind (R,θ) i.e. the probability with respect to the unity of surface when one imagines this surface to be small. If we put for simplification:

$$h'^2 + h^2 = 2p, \quad h'^2 - h^2 = 2q, \quad R_0^2 \left(p - q \cos 2\left(\alpha - \beta\right)\right) = \mu$$

$$\left(p - q \cos 2\left(\theta - \beta\right)\right) = \nu, \quad s^2 = R_0^2 \left(p^2 + q^2 - 2pq \cos 2\left(\alpha - \beta\right)\right)$$

$$s \cos\left(\theta - \varphi\right) = \lambda \quad \operatorname{tang} \varphi = \frac{p \sin \alpha + q \sin\left(\alpha - 2\beta\right)}{p \cos \alpha - q \cos\left(\alpha - 2\beta\right)},$$

then (2) takes the form:

$$\frac{\sqrt{p^2 - q^2}}{\pi} e^{-R^2\nu + 2R\lambda - \mu} RdRd\theta. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If here we put:

$$R^2\nu - 2R\lambda + \mu = c \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

then it follows out of the above formulated definition that the specific probability of all observations lying on the circumference of the excentric ellipse (6) is the same and equal to:

$$\frac{\sqrt{p^2 - q^2}}{\pi} e^{-c}.$$

The probability that the velocity of the wind does not surpass the value R_c expressed by (6) in function of θ, in other words the number of observations which are to lie within the area of the ellipse, is found by integrating (5), first with respect to R between the limits R_c and 0, then with respect to θ between 2π and 0.

For the simple case $R_0 = 0$, so also $\mu = 0$ and $\lambda = 0$, the first integration gives immediately

$$\frac{\sqrt{p^2-q^2}}{\pi} \cdot \frac{1-e^{-c}}{2v}$$

and as

$$\frac{\sqrt{p^2-q^2}}{2\pi} \int_0^{2\pi} \frac{d\theta}{v} = 1,$$

the probability to be found becomes simply:

$$1-e^{-c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and the number of observations lying inside the circumference of the ellipse (6):

$$N(1-e^{-c}).$$

This amount remaining the same whether we regard the ellipse (6) from the excentric origin or from the centre, i.e. for $R_0 = 0$, if with the integration the limits are changed correspondingly, the expression (7) must also be accurate when R_0 is not equal to zero and must thus hold in general.

Indeed, an other simplification, namely $q = 0$ (which is applicable to the results for Falmouth) leads to a set of definite integrals, which can be evaluated and which confirm this conclusion.

Amongst the series of ellipses represented by (6) two are remarkable; if we assign to c the value 0.5, then on account of (1) the half axes of the ellipse become equal to the greatest and smallest projections M and M' of the mean velocities, so that the ellipse (6) then represents what we might call the *specific* or *typical windellipse,* thus a kind of windrose, in which the characteristic qualities of the wind-distribution under consideration inmediately become conspicuous.

The radius vector R_m drawn to an arbitrary point in the circumference is given in the direction determined by that choice by the equation:

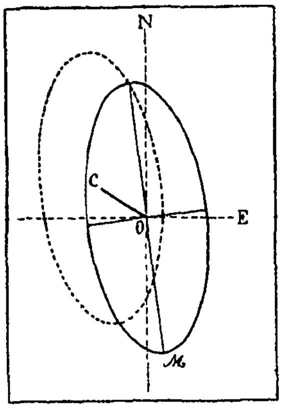
$$2\,R_m^2\,v - 4\,R_m\,\lambda + 2\,\mu - 1 = 0.$$

The probability that a velocity does not surpass this value is:

$$1 - e^{-1/2} = 0.39347.$$

So among a thousand observations there will be 393 lying inside this typical ellipse whilst the specific probability of each of the velocities R_m is:

$$0.6065 \frac{\sqrt{p^2-q^2}}{\pi}.$$



In the given diagram such a typical windellipse is represented for Bergen in the month of June by the dotted line; the vector OC represents here the constant part (R_0, α), the half axes are equal to M and M', and the angle $NOM = \beta$; one millimeter corresponds to $^3/_{20}$ Beaufort scale-value or to $^3/_{20} \times 1.83 = 0.275$ meter a second.

If necessary this diagram might be amplified with two circles, one of a radius

$$\sqrt{M^2 + M'^2},$$

representing the mean monthly wind velocity corrected for the constant part, the other described with radius

$$\sqrt{M^2 + M'^2 + (M_x)^2 + (M_y)^2},$$

which is according to (4) a measure for the mean total velocity, corresponding to the square root of the quantity V^2 of the tables III and IV.

An other remarkable ellipse which might be called the *probable windellipse* is obtained by requiring half of the observations to lie within its dominion; we have then to determine c in such a way that

$$1 - e^{-c} = {}^1/_2 \quad , \quad c = 0.6932,$$

so that the axes of this ellipse are

$$\sqrt{2c} = \sqrt{2} \times 0.8326 = 1.177$$

times longer than those of the typical windellipse; the number 0.8326 is a quantity known in the theory of errors in the plane.

7. The frequency of the windvelocities, setting aside the direction, cannot be represented in a finite form; we can arrive at a form serviceable for comparison with the observation by writing (5) thus:

$$\frac{\sqrt{p^2-q^3}}{\pi} e^{-\mu} \,.\, e^{-\nu R^2} \,.\, e^{-R^2(\cdot-\mu)+2R)} R\,dR\,d\theta, \quad . \quad . \quad . \quad (8)$$

by developing the last exponential factor and then by expressing the powers and products of cosines in cosines of multiples.

It is clear that when integrating (8) with respect to θ from 2π to 0 only those terms are left which are independent of θ and which appear with the common factor 2π.

The expression to be found for the probability that a velocity lies between the limits R and $R+dR$ then becomes:

$$2\sqrt{p^2-q^2}\,.\,e^{-\mu}\,.\,e^{-pR^2}(1+a_1R^2+a_2R^4\;\ldots)\,R\,dR\,,\quad\ldots\quad(9)$$

where:

$$a_1 = s^2,$$

$$a_2 = q_2/2^2 \quad + qs^2/2!\ \cos 2\,(\varphi-\beta) + s^4/(2!)^2,$$

$$a_3 = q^2s^2/2^2 + qs^4/3!\ \cos 2\,(\varphi-\beta) + s^6/(3!)^2.$$

For Falmouth, where as was noticed above q can be put equal to nought these coefficients become simply:

$$a_n = s^{2n}\Big/\left(\frac{n!}{2}\right)^2,$$

and farther

$$s = pR_0\;,\;\mu = pR_0^{\,2}\;,\;\nu = p\;,\;\varphi = \alpha\;,\;\lambda = pR_0\cos(\theta-\alpha).\quad(10)$$

In practice it will frequently be only necessary to calculate a few of these coefficients; if we put:

$$q/p = \varepsilon,$$

the integration of (9) between the limits m and 0 leads to the expression:

$$\left.\begin{aligned}&\sqrt{1-\varepsilon^2}\,e^{-\mu}\times\\&(1-e^{-pm^2})\,.\left(1+\frac{a_1}{p}+\frac{2!a_2}{p^2}+\frac{3!a_3}{p^3}+\ldots\right)\\&-\frac{pm^2e^{-pm^2}}{1!}\left(\frac{a_1}{p}+\frac{2!a_2}{p^2}+\ldots\ldots\ldots\right)\\&-\frac{p^2m^4e^{-pm^2}}{2!}\left(\frac{2!a_2}{p^2}+\ldots\ldots\ldots\ldots\right)\end{aligned}\right\}\;.\quad(11)$$

As for $m=\infty$ this expression must become equal to unity, we have:

$$1+\frac{a_1}{p}+\frac{2!a_2}{p^2}\;\ldots\ldots = \frac{e^{\mu}}{\sqrt{1-\varepsilon^2}},$$

or, for the case $q=0$, (11) becomes:

$$\left.\begin{array}{l} 1-e^{-pm^2} \\ -\dfrac{pm^2e^{-pm^2}}{1!}(1-e^{-\mu}) \\ -\dfrac{p^2m^4e^{-pm^2}}{2!}\left(1-e^{-\mu}-\dfrac{a_1}{p}e^{-\mu}\right) \text{ etc.} \end{array}\right\}, \quad . \quad . \quad . \quad (12)$$

from which is immediately evident that in many cases the three first terms are sufficient, so that then the calculation of the coefficients can be entirely avoided, or at most only a_1 must be taken into account; for generally μ is small, so that already

$$1-e^{-\mu}$$

will be a small quantity. If q is not small the calculation becomes rather tedious.

8. To find expressions for the quantities $\overline{V}$ and $\overline{V^2}$, the mean velocity and the mean square of the velocity independent of the sign, we have to multiply (9) successively by R and R^2 and to integrate between the limits ∞ and 0 which, with the well known fundamental equation, leads to the expressions:

$$\left.\begin{array}{l} 1 = A\left(1+\dfrac{2a_1}{2p}+\dfrac{2\ 4a_2}{(2p)^2}+\dfrac{2.4.6a_3}{(2p)^3}+\dots\right) \\ \overline{V} = \dfrac{A}{2}\sqrt{\dfrac{\pi}{p}}\left(1+\dfrac{3a_1}{2p}+\dfrac{3.5a_2}{(2p)^2}+\dfrac{3.5.7.a_3}{(2p)^3}+\dots\right) \\ \overline{V^2} = \dfrac{2A}{2p}\left(1+\dfrac{4a_1}{2p}+\dfrac{4.6a_2}{(2p)^2}+\dfrac{4.6.8a_3}{(2p)^3}+\dots\right) \\ A = \sqrt{1-\varepsilon^2 e^{-\mu}}. \end{array}\right\} . \quad (13)$$

9. For the calculation of the frequency of the directions independent of the velocity we have first to integrate (5) with respect to R between the limits ∞ and 0 and then with respect to θ between the desired limits θ; the mean velocity as function of the direction is found by the application of the same operation to (5) after multiplication by R. It is then easy to give to a frequency-formula found in this way the form of a Fourier series. For brevity we treat here only the case that $q=0$ and the angle-limits are π to 0.

By putting

$$R=\varrho+\frac{\lambda}{\nu}$$

we get (5) reduced to the form:

$$\frac{\sqrt{p^2-q^2}}{\pi} e^{-\nu+\lambda^2/\nu} \int_{-\lambda/\nu}^{\infty} e^{-\nu\rho^2} \left(\rho + \frac{\lambda}{\nu} \right) d\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $q = 0$, so that the formulae (10) hold good, we then find for the desired frequencies in the two easterly quadrants

$$\tfrac{1}{2} + \frac{1}{\sqrt{\pi}} \int^{R_0 \sqrt{p} \sin \alpha} e^{-t^2} \, dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

From this formula it is evident in what way and in what degree the asymmetry of the distribution is dependent of R_0, α and p.

10. The application of the given criteria has been made for Falmouth and the four seasons:

Winter: December, January, February, number 8384,
$p = 0.00258$, $q = 0.00004$
$R_0 = 5.22$, $\alpha = 222°8'$

Spring: March, April, May, number 8949,
$p = 0.00298$, $q = 0.00028$
$R_0 = 2.21$, $\alpha = 250°25'$

Summer; June, July, August, number 9229,
$p = 0.00485$, $q = 0.00029$
$R_0 = 5.60$, $\alpha = 251°22'$

Autumn: September, October, November, number 9254,
$p = 0.00313$, $q = 0.00004$
$R_0 = 3.80$, $\alpha = 239°16'$

For each series the number of observations is reduced to 10.000 and everywhere we have put $q = 0$, the calculated values are accordingly accurate as far as the fourth decimal.

In Table V we have compared the observed frequencies of wind-velocities independent of direction with those calculated according to formula (12), from which it is evident that the differences have a clearly systematic course. Just as is the case with all series of errors the number of the observed small velocities is larger than would agree with the normal distribution. The differences together amount in summer to about 10°/₀, in winter to 15°/₀.

In the calculation of the frequencies of the directions independent of the velocity, the observations regarded as calms — and to these are reckoned in the English records all velocities less than 4 miles an hour — have been distributed proportionally to the frequencies of direction; furtheron the frequencies North and South are assigned for one half to the eastern and western quadrants.

As is evident from the following table also in this comparison systematic differences appear; in all seasons the observed frequencies in the western quadrant are greater than the calculated ones, so that an increase of the constant part R_0 to which this uneven distribution can be attributed, would improve the correspondence.

TABLE VI.

Frequencies of winddirections at Falmouth

for 10.000 observations.

		Observed	Calculated	Difference
Winter	E. quadr.	3709	4006	— 297
	W. „	6291	5994	+ 297
Spring	E. quadr.	4037	4351	— 214
	W. „	5963	5649	+ 214
Summer	E. quadr.	2619	3009	— 390
	W. „	7381	6991	+ 390
Autumn	E. quadr.	3453	3980	— 527
	W. „	6547	6020	+ 527

TABLE V. Frequencies of windvelocities at Falmouth.
For 10.000 observations.

	WINTER			SPRING			SUMMER			AUTUMN		
Miles an hour	Observ.	Calculat.	Difference	Observ.	Calculat.	Difference	Observ.	Calculat.	Difference	Observ.	Calculat.	Difference
$0 - 4^5$	760	477	+ 283	756	569	+ 187	945	810	+ 135	936	588	+ 348
$4^5 - 9^5$	1871	1482	+ 389	2073	1759	+ 314	2610	2336	+ 274	2126	1779	+ 377
$9^5 - 14^5$	1853	2026	— 173	2120	2279	— 159	2538	2728	— 190	2144	2304	— 160
$14^5 - 19^5$	1701	2030	— 329	1875	2120	— 245	1868	2133	— 265	1818	2110	— 292
$19^5 - 24^5$	1466	1650	— 184	1355	1558	— 203	1164	1225	— 61	1281	1564	— 283
$24^5 - 29^5$	967	1125	— 158	917	939	— 22	544	5[illegible]4	+ 10	792	920	— 128
$29^5 - 34^5$	680	656	+ 24	493	473	+ 20	224	178	+ 46	459	464	— 5
$34^5 - 39^5$	369	331	+ 38	250	201	+ 49	80	45	+ 35	258	179	+ 79
$39^5 - 44^5$	199	144	+ 55	114	72	+ 42	20	9	+ 11	116	66	+ 50
$44^5 - 49^5$	94	55	+ 39	31	22	+ 9	6	2	+ 4	37	20	+ 17
$49^5 - 54^5$	32	18	+ 14	15	6	+ 9	0	—	—	23	5	+ 18
$54^5 - 59^5$	8	5	+ 3	1	2	— 1	1	—	+ 1	10	1	+ 9
$59^5 -$	—	1	— 1	—	—	—	—	—	—	—	—	—