Mathematics. — *"On pencils of algebraic surfaces."* By Prof. JAN DE VRIES.

1. Let a pencil (F^n) of surfaces F^n of order n be given, intersecting in the base-curve σ.

The principal tangents in a point S of σ to the surfaces of (F^n) form a cubic cone having the tangent s to σ for edge.

For, if (F^n) is indicated by

$$a_x^n + \lambda b_x^n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if y_k are the coordinates of S, the substitution $x_k = y_k + \varrho z_k$ furnishes in connection with $a_y^n = 0$ and $b_y^n = 0$ for a point Z on a principal tangent the conditions

$$a_y^{n-1} a_z + \lambda b_y^{n-1} b_z = 0 \quad , \quad a_y^{n-2} a_z^2 + \lambda b_y^{n-2} b_z^2 = 0 \, ,$$

so that the locus of the principal tangents touching in S has as equation

$$a_y^{n-1} b_y^{n-2} a_z b_z^2 - a_y^{n-2} b_y^{n-1} a_z^2 b_z = 0 \quad . \quad . \quad . \quad . \quad (2)$$

If Z is a fixed point, Y a variable one, this equation represents a surface of order $(2n-3)$ determining on σ the points S which are points of contact of principal tangents through Z.

The principal tangents in points of the base-curve form a congruence of order $n^2(2n-3)$ and of class $3n^2$.

The inflexional tangents of a pencil of plane curves c^n enveloping a curve of class $3n(n-2)$, *the complex of rays of the principal tangents* of F^n is of order $3n(n-2)$.

2. A principal tangent t_3 in S becomes four-pointed tangent t_4, if for a point Z lying on it the relation $a_y^{n-3} a_z^3 + \lambda b_y^{n-3} b_z^3 = 0$ holds good. So the tangents t_4 touching in S belong to the biquadratic cone

$$a_y^{n-1} b_y^{n-3} a_z b_z^3 - a_y^{n-3} b_y^{n-1} a_z^3 b_z = 0 \quad . \quad . \quad . \quad . \quad (3$$

As the cones (2) and (3) have the tangent s represented by

$$a_y^{n-1} a_z = 0 \quad , \quad b_y^{n-1} b_z = 0 \quad , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in common, the point S will be the point of contact of *eleven four-pointed tangents.*

For $a_y^n = 0$ and $b_y^n = 0$ the equations (2) and (3) represent the figure formed by the surface of tangents (s) of σ and the scroll τ_4 of the right lines t_4 having their points of contact on σ.

To determine the order of τ_4 I search for the number of points of intersection of the indicated figure with the right line $z_3 = 0$, $z_4 = 0$. Substitution in (2) and in (3) and elimination of z_1 and z_2 gives an equation containing the coefficients of (2) and (3) successively in the orders 4 and 3. Hence the resultant in the coordinates y is of order $4(2n - 3) + 3(2n - 4)$ or $14n - 24$.

So the number of points of intersection is $2n^2(7n - 12)$.

Applying the same treatment to (4) I find for the order of (s) the well-known number $2n^2(n - 1)$.

The four-pointed tangents having their points of contact on the base-curve σ form a scroll of order $2n^2(6n - 11)$, on which σ is elevenfold.

For $n = 3$ this scroll passes into the locus of the right lines on the surfaces of a pencil (F^3), thus into the scroll of the trisecants of σ^9 which is of order 42. For $2n^2(6n - 11)$ we now find 126, corresponding to the fact, that each trisecant does duty as right line t_4 for three points S.

On each surface F^n the points of contact P_4 of four-pointed tangents t_4 form a curve of order $n(11n - 24)$. As σ is evidently an elevenfold curve of the locus of the points of contact P_4 belonging to the F^n of the pencil this locus has in common with every F^n a locus of order $n(11n - 24) + 11n^2$.

The points of contact of four-pointed tangents form a surface of order $2(11n - 12)$.

For $n = 3$ we find the scroll of trisecants of order 42.

3. As the tangents t_3 passing through a point Z form a cone of order $3n(n - 2)$ and two principal tangents have their point of contact in Z the locus of the points of contact P_3 of the right lines t_3 containing Z is a curve of order $3n(n - 2) + 2$ with double point in Z.

Each of those right lines t_3 cuts the surface F^n osculated by it in $(n - 3)$ points Q more. The locus of the points Q is a curve of order $3n(n - 2)(n - 3) + (n^2 + 2)(n - 3)$ or $2(n - 3)(n - 1)(2n - 1)$. For, through Z pass $(n^2 + 2)(n - 3)$ tangents t_3, osculating the surface indicated by Z in an other point [1]).

To find the number ε of the coincidences of P_3 with Q, I make use of the well-known formula

$$\varepsilon = p + q - g,$$

which appears when the pairs of points P, Q are projected by a pencil of planes. Each point P belonging to $(n-3)$ pairs we have

[1]) See CREMONA—CURTZE, Oberflächen, p. 66.

$p = (3n^2 - 6n + 2)(n - 3)$. Farther more $q = 2(n-3)(n-1)(2n-1)$, whilst the number of right lines PQ resting on an axis is of course equal to $3n(n-2)(n-3)$. So $\varepsilon = 2(n-3)(2n^2 - 3n + 2)$; this is also the number of four-pointed tangents through a given point a.

The number of right lines t_4 in a given plane is equal to the number of points of undulation on the curves c^n of a pencil; this number I have determined in a preceding paper [1]).

The four-pointed tangents form a congruence of order

$2(n-3)(2n^2 - 3n + 2)$ *and of class* $\frac{9}{2}(n-3)(n^3+n^2-8n+4)$.

4. If we wish to apply the above-mentioned formula of coincidence to the pairs of points of intersection Q, Q' on the right lines t_3 through Z we have to substitute $p = q = 2(n-3)(n-1)(2n-1)(n-4)$ and $g = 3n(n-2)(n-3)(n-4)$. For each point Q belongs to $(n-4)$ pairs and each right line t_3 bears $(n-3)(n-4)$ pairs. We then find $\varepsilon = (n-3)(n-4)(5n^2 - 6n + 4)$, i. e. the number of tangents $t_{3,2}$ through the point Z.

In the above-mentioned paper I have determined the number of right lines having with a curve of a pencil (c^n) a three-pointed and at the same time a two-pointed contact.

The two-three-pointed tangents form a congruence of order $(n-3)(n-4)(5n^2 - 6n + 4)$ *and of class*

$$\frac{1}{2}(n-4)(n-3)^2(10n^4 + 35n^3 - 21n^2 - 80n + 20).$$

5. Each principal tangent t_3 having its point of osculation in a point S of the base-curve bears still $(n-3)$ points of intersection Q with the surface F^n osculated by it. As S is point of contact of 11 four-pointed tangents the locus of the points Q will have an elevenfold point in S. As an arbitrary plane through S evidently contains $3(n-3)$ points Q (§ 1) the order of the curve (Q) is equal to $(3n + 2)$.

When applying the formula $\varepsilon = p + q - g$ to the pairs Q, Q' which the cubic cone with vertex S bears, we have to put $p = q = (3n+2)(n-4)$ and $g = 3(n-3)(n-4)$. Then we get $\varepsilon = (n-4)(3n+13)$. So this is the number of tangents $t_{3,2}$ for which the point of osculation lies in S.

In other words, σ is an $(n-4)(3n+13)$-fold curve on the locus $[R_3]$ of the *points of osculation* of tangents $t_{3,2}$ to surfaces of (F^n). Now the points of osculation of the right lines $t_{3,2}$ of an F^n form a

1) "On linear systems of algebraic plane curves", (Proceedings, April 22, 1905.)

curve of order $n(n-4)(3n^2+5n-24)$[1]). So $[R_3]$ has in common with an F^n of the pencil a curve of order

$$n(n-4)(3n^2+5n-24)+n^2(n-4)(3n+13)=n(n-4)(6n^2+18n-24).$$

The points of osculation of the three-two-pointed tangents of (F^n) *form a surface of order* $6(n-1)(n-4)(n+4)$.

6. To determine the order of the cone formed by the double tangents of (F^n) of which a point of contact in S lies on the base-curve σ, we notice that the tangent s in S to σ is intersected by a pencil in an involution of order $(n-2)$. Its $2(n-3)$ double points are points of contact of double tangents touching in S too. So s is a $2(n-3)$-fold edge of the indicated cone.

In each plane ϕ through s we can draw out of S $n(n-1)-6$ tangents to the curve of intersection of ϕ with the surface F^n touching ϕ in S. From this ensues that the indicated cone is of order $(n-3)(n+4)$.

The locus of the second points of contact R_2 of the edges of this cone has evidently in S an elevenfold point, where it is touched by the eleven right lines t_4. So the curve (R_2) is of order

$$(n-3)(n+4)+11=n^2+n-1.$$

Every edge of the cone intersects the surface doubly touched by it in $(n-4)$ points V more. The locus of these points passes $(n-4)(3n+13)$ times through S, where it is touched by the right lines $t_{3,2}$ osculating in S. As each plane through S bears moreover $(n^2+n-12)(n-4)$ points V, the curve (V) is of order

$$(n-4)(n^2+4n+1).$$

Now the number of coincidences of R_2 with V can be determined again by means of the formula $\varepsilon=p+q-g$. We find

$$\varepsilon=(n^2+n-1)(n-4)+(n-4)(n^2+4n+1)-(n-3)(n+4)(n-4)=$$
$$=(n-4)(n^2+4n+12).$$

This is the number of tangents $t_{3,2}$, of which the *point of contact* lies in S, thus at the same time the multiplicity of the base-curve on the surface $[R_2]$ of the points of contact of surfaces of pencils with right lines $t_{3,2}$. Taking into consideration, that the points of contact R_2 form on the surface F^n a curve of order

$$n(n-2)(n-4)(n^2+2n+12)$$

we find that $[R_2]$ has with F^n an intersection of order

$$n(n-2)(n-4)(n^2+2n+12)+n^2(n-4)(n^2+4n+12).$$

[1]) See inter alia my paper: "Some characteristic numbers of an algebraic surface." Proceedings, April 22nd, 1905).

The points of contact of the three-pointed tangents of (F^n) *form a surface of order* $2(n-4)(n^3+2n^2+10n-12)$.

7. Through the tangent s in S to σ we can make to pass four tangent planes to the cubic cone of the principal tangents (§ 1). So S is a parabolic point on four surfaces of the pencil. Therefore σ is a fourfold curve on the locus of the parabolic points.

As the parabolic points of an F^n lie on a curve of order $4n(n-2)$ the locus under consideration is cut by each of the surfaces F^n in a curve of order $4n(n-2)+4n^2=8n(n-1)$.

The locus of the parabolic points of the surfaces of a pencil (F^n) *is a surface of order* $8(n-1)$.

Chemistry. — *"On the shape of the plaitpoint curve for mixtures of normal substances."* (Second communication). By J. J. VAN LAAR. (Communicated by Prof. H. A. LORENTZ).

1. In a previous paper [1]), starting from VAN DER WAALS' equation of state, in which b is assumed to be independent of v and T, I have found for the equation of the *spinodal curves* at successive temperatures (l. c. p. 690):

$$RT=\frac{2}{v^3}\left[x(1-x)\theta^2+a(v-b)^2\right], \quad . \quad . \quad . \quad . \quad (1)$$

and for that of the plaitpoint curve in its v, x projection (l.c. p. 695):

$$x(1-x)\theta^3\left[(1-2x)v-3x(1-x)\beta\right]+\sqrt{a}(v-b)^2\left[3x(1-x)\theta(\theta-\beta\sqrt{a})+a(v-b)(v-3b)\right]=0 \quad . \quad . \quad . \quad . \quad (2)$$

In this $\theta=\pi+\alpha(v-b)$, $\pi=b_1\sqrt{a_2}-b_2\sqrt{a_1}$, $\alpha=\sqrt{a_2}-\sqrt{a_1}$, and $\beta=b_2-b_1$.

The equations (1) and (2) hold for the so-called *symmetrical* case, where not only $b_{12}={}^1/_2(b_1+b_2)$ is assumed, but also $a_{12}=\sqrt{a_1a_2}$. These hypotheses lead to:

$$b=(1-x)b_1+xb_2 \quad ; \quad a=[(1-x)\sqrt{a_1}+x\sqrt{a_2}]^2.$$

The equation (1) had been given already before by VAN DER WAALS in implicit form [2]), for after some reduction his general equation

[1]) These Proc. April 22, 1905, p. 646—657.

[2]) Cont. II, p. 45; Arch. Néerl. **24**, p. 52 (1891).