here to trace how and on what grounds this name came to be general. I intend to explain this afterwards at some length.

Finally one remark more. The nucleus of the Hondsrug in South Drente is rightly said to be of a fluviatile nature. Yet it may be asked whether the boulder-sand, to keep to this, immediately rests on it and whether nothing can be observed there of formations known as stratified mixed and stratified glacial diluvium. This would indeed be very striking and in fact this is not always the case, though I must acknowledge that I have found in the discussed part of the Hondsrug only with great difficulty some profiles in which somewhat acute bounding lines may be observed. I must however put off this discussion. I have mentioned here only so much of my own observations as was strictly necessary; in a complete treatise of it I hope to have an opportunity to enter into the question of the origin of the Hondsrug more in particulars.

Groningen, Min.-Geol. Institute, June 6, 1905.

Astronomy. — "*Approximate formulae of a high degree of accuracy for the ratio of the triangles in the determination of an elliptic orbit from three observations* II." By J. WEEDER. (Communicated by Prof. H. G. VAN DE SANDE BAKHUYZEN).

In connection with my paper on the same subject read on 22 April 1905 I now intend to derive simple approximate formulae for the ratio of the triangles, which contain the 3 times of observation and the heliocentric distances belonging to them, and include the terms of the 5th order with respect to the intervals of time, it being easy to add, if necessary, those of the sixth order. The same problem has been treated by P. HARZER, and in the developments at which he arrived he attained a much higher degree of precision [1]). Nevertheless it appears to me that his publication does *not* render mine superfluous because of the different methods of the treatment and the conciseness of my results.

After the method, followed by GIBBS, to derive his fundamental equation, we find with satisfactory approximation a general relation between the values of a function $F(\tau)$ at the three instants, its second derivatives with regard to the time $\ddot{F}(\tau)$ at the same instants

[1]) P. HARZER, Ueber die Bestimmung und Verbesserung der Bahnen von Himmelskörpern nach drei Beobachtungen. Met einem Anhange unter Mithilfe von F. RISTENPART und W. EBERT berechneter Tafeln. Leipzig 1901. Publication der Sternwarte Kiel XI.

and the two intervals of time τ_3 and τ_1. The value namely, of the following expression

$$\tau_1\left(F_1+\ddot{F}_1\frac{\tau_1^2-\tau_2\tau_3}{12}\right)-\tau_2\left(F_2+\ddot{F}_2\frac{\tau_2^2+\tau_1\tau_3}{12}\right)+\tau_3\left(F_3+\ddot{F}_3\frac{\tau_3^2-\tau_1\tau_2}{12}\right)$$

where $\tau_2=\tau_1+\tau_3$, is of the 6th order with respect to the intervals. I use the letters C_1, C_2 and C_3 to designate the multipliers of the second derivatives in this expression and put

$$C_1=\frac{\tau_1\tau_2\tau_3-\tau_1^3}{12},\ C_2=\frac{\tau_1\tau_2\tau_3+\tau_2^3}{12},\ C_3=\frac{\tau_1\tau_2\tau_3-\tau_3^3}{12}.$$

Neglecting the terms of the 6th order we then have for an arbitrary function of the time, the relation

$$\tau_1F_1-C_1\ddot{F}_1-\tau_2F_2-C_2\ddot{F}_2+\tau_3F_3-C_3\ddot{F}_3=0\ \ .\ \ .\ \ (IV)$$

provided this function and its first four derivatives be continuous and finite within the interval τ_2.

Applying this formula to the heliocentric distance r and to r^2, I obtain approximate expressions for the semi-parameter p, and the semi-axis major a of the elliptic orbit. By eliminating p, from the two well-known differential equations $\ddot{r}r^3+r=p$ and $\dot{r}^2=\frac{2}{r}-\frac{p}{r^2}-\frac{1}{a}$, I find a differential equation which may be easily reduced to $\frac{d^2}{d\tau^2}(r^2)=2\left(\frac{1}{r}-\frac{1}{a}\right)$. According to these relations $\ddot{F}=\frac{p-r}{r^3}$ belongs to $F=r$, and $\ddot{F}=2\left(\frac{1}{r}-\frac{1}{a}\right)$ to $F=r^2$.

If as before I put z for $1/r^3$, the substitution of $F=r$ in formula IV yields the following equation to determine p,

$$\tau_1r_1-\tau_2r_2+\tau_3r_3-C_1z_1(p-r_1)-C_2z_2(p-r_2)-C_3z_3(p-r_3)=0$$

whence:

$$p=\frac{(\tau_1+C_1z_1)r_1-(\tau_2-C_2z_2)r_2+(\tau_3+C_3z_3)r_3}{C_1z_1+C_2z_2+C_3z_3}\ \ .\ \ .\ \ (V)$$

Through the substitution of $F=r^2$ in IV I obtain the equation

$$\tau_1r_1^2-\tau_2r_2^2+\tau_3r_3^2-2C_1\left(\frac{1}{r_1}-\frac{1}{a}\right)-2C_2\left(\frac{1}{r_2}-\frac{1}{a}\right)-2C_3\left(\frac{1}{r_3}-\frac{1}{a}\right)=0$$

whence

$$\frac{1}{a}=\frac{-\tau_1r_1^2+\tau_2r_2^2-\tau_3r_3^2+2\left(\frac{C_1}{r_1}+\frac{C_2}{r_2}+\frac{C_3}{r_3}\right)}{2(C_1+C_2+C_3)}\ \ .\ \ .\ \ (VI)$$

The terms neglected in these expressions for p and $\frac{1}{a}$ are of the 3d order with respect to the intervals of time.

I shall now proceed to show how we can avail ourselves of these values for p and $\frac{1}{a}$ for the calculation of the ratios of the triangles.

In my previous paper I have demonstrated that the area of the triangle PZP_1 considered as a function of $\tau = k(t - t_1)$ satisfies the differential equation $\ddot{F} + z\,F = 0$. The same differential equation is satisfied by the area of the triangle P_2ZP, considered as a function of $\tau = k(t_2 - t)$. The two areas may, according to MAC LAURIN be expressed in series of the ascending powers of τ. If the variable τ takes the value $k(t_2 - t_1) = \tau_3$, the two triangles become equal to P_2ZP_1; therefore it will be possible to obtain a new expansion in series for double the area $\Delta\, P_2ZP_1$, by putting in the sum of the two former series $\tau = \tau_3$. From this new series we can easily remove the terms with the even powers of τ.

According to this plan I give here first some higher derivatives of the function F, expressed in F, $\dot{F}$, z and derivatives of z with respect to the same variable.

$$F^{III} = -\dot{z}\,F - z\dot{F}$$

$$F^{IV} = (z^2 - \ddot{z})\,F - 2\dot{z}\,\dot{F}$$

$$F^{V} = (4z\dot{z} - z^{III})\,F + (z^2 - 3\ddot{z})\,\dot{F}$$

$$F^{VI} = (-z^3 + 4\dot{z}^2 + 7z\ddot{z} - z^{IV})\,F + 2\,(3z\dot{z} - 2z^{III})\,\dot{F}$$

$$F^{VII} = (\,.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\,)\,F + (13z\ddot{z} + 10\dot{z}^2 - 5z^{IV} - z^3)\,\dot{F}.$$

For $\tau = 0$, the value of the function $\frac{\text{triangle } PZP_1}{\sqrt{p}} = F\,[k(t - t_1)] = F(\tau)$ and that of its first derivative is known, viz. $F_0 = 0$ and $\dot{F}_0 = +\frac{1}{2}$.

The above mentioned expansion in series for ΔPZP_1 is therefore:

$$\frac{\Delta PZP_1}{\sqrt{p}} = \frac{1}{2}\,\frac{\tau}{1} - \frac{1}{2}\,z_1\,\frac{\tau^3}{3!} - \dot{z}_1\,\frac{\tau^4}{4!} + \frac{1}{2}\,(z_1^{\,2} - 3\ddot{z}_1)\,\frac{\tau^5}{5!} + (3z_1\dot{z}_1 - 2z_1^{\,III})\,\frac{\tau^6}{6!} +$$

$$+ \frac{1}{2}\,(13z_1\ddot{z}_1 + 10\dot{z}_1^{\,2} - 5z_1^{\,IV} - z_1^{\,3})\,\frac{\tau^7}{7!} + \int_0^{\tau} \frac{u^7}{7!}\,F^{VIII}\,(\tau - u)\,du\,.$$

The function $\frac{\Delta P_2ZP}{\sqrt{p}} = G\,[k\,(t_2 - t)] = G\,(r)$ and its derivative also take for $t = t_2$ or $\tau = 0$ the values $G_0 = 0$ and $\dot{G}_0 = +\frac{1}{2}$, and so for this function, because it satisfies the differential equation $\ddot{G} + z\,G = 0$, the same expansion holds as for $F(\tau)$, but while in the series for ΔPZP_1 the derivatives are taken with regard to increasing time, those in that for ΔP_2ZP must be considered with regard to decreasing time. If we make use of the symbols $\dot{z}$, $\ddot{z}$, z^{III} etc. to denote derivatives of z with regard to increasing time, the signs of the odd derivatives of z in the expansion for ΔP_2ZP must be reversed. Hence we obtain for ΔP_2ZP:

$$\frac{\Delta P_2ZP}{\sqrt{p}} = \frac{1}{2}\frac{\tau}{1} - \frac{1}{2} z_2 \frac{\tau^3}{3!} + \dot{z}_2 \frac{\tau^4}{4!} + \frac{1}{2}(z_2{}^2 - 3\ddot{z}_2)\frac{\tau^5}{5!} - (3z_2\dot{z}_2 - 2z_2{}^{\text{III}})\frac{\tau^6}{6!} +$$

$$+ \frac{1}{2}(13z_2\ddot{z}_2 + 10\dot{z}_2{}^2 - 5z_2{}^{\text{IV}} - z_2{}^3)\frac{\tau^7}{7!} + \int_0^\tau \frac{u^7}{7!} G^{\text{VIII}}(\tau - u)\,du$$

and by summation of the two series, for $\tau = \tau_3$

$$2\,\frac{\Delta P_2ZP_1}{\sqrt{p}} = \frac{\tau_3}{1} - \frac{z_1 + z_2}{2}\frac{\tau_3{}^3}{3!} + (\dot{z}_2 - \dot{z}_1)\frac{\tau_3{}^4}{4!} +$$

$$+ \left\{\frac{1}{2}(z_2{}^2 - 3\ddot{z}_2) + \frac{1}{2}(z_1{}^2 - 3\ddot{z}_1)\right\}\frac{\tau_3{}^5}{5!} - \left\{(3z_2\dot{z}_2 - 2z_2{}^{\text{III}}) - (3z_1\dot{z}_1 - 2z_1{}^{\text{III}})\right\}\frac{\tau_3{}^6}{6!} +$$

$$+ \left[\frac{13z_2\ddot{z}_2 + 10\dot{z}_2{}^2 - 5z_2{}^{\text{IV}} - z_2{}^3}{2} + \frac{13z_1\ddot{z}_1 + 10\dot{z}_1{}^2 - 5z_1{}^{\text{IV}} - z_1{}^3}{2}\right]\frac{\tau_3{}^7}{7!} +$$

$$+ \int_0^{\tau_3} \frac{u^7}{7!}\left\{F^{\text{VIII}}(\tau - u) + G^{\text{VIII}}(\tau - u)\right\}du.$$

It appears that in this formula the terms with even powers of τ_3 can be transformed into series of terms with the higher odd powers of τ_3. In order to do this I derive an expansion in series by which this aim is reached in a general manner for the difference $f(y) - f(x)$, f being an arbitrary function which between x and y does not show singularities. Let here τ be put for $y - x$, and m for $\frac{y + x}{2}$ then

$$f(y) - f(x) = \int_{-\tau/2}^{+\tau/2} f'(m + u)\,du$$ and after integration by parts:

$$f(y) - f(x) = \frac{\tau}{2}\left[f'(y) + f'(x)\right] - \int_{-\tau/2}^{+\tau/2} u\,f''(m + u)\,du.$$

As $\int_{-\tau/2}^{+\tau/2} u f''(m)\, du = 0$ and $f''(m+u) - f''(m) = \int_0^u f^{\mathrm{III}}(m+v)\, dv$,

we may write instead of $\int_{-\tau/2}^{+\tau/2} u f''(m+u)\, du$, the double integral $\int_{-\tau/2}^{+\tau/2} u\, du \int_0^u f^{\mathrm{III}}(m+v)\, dv$ which by reversing the order of integration is transformed into $\int_{-\tau/2}^{+\tau/2} f^{\mathrm{III}}(m+v)\, dv \int_v^{+\tau/2} u\, du$.

I now proceed to integrate with respect to u, and so we obtain the following relation:

$$f(y) - f(x) = \frac{\tau}{2}[f'(y) + f'(x)] - \frac{1}{8}\int_{-\tau/2}^{+\tau/2} f^{\mathrm{III}}(m+v)(\tau^2 - 4v^2)\, dv.$$

The operations may be repeated and by doing so we shall find:

$$f(y) - f(x) = \frac{\tau}{2}[f'(y) + f'(x)] - \frac{\tau^3}{24}[f^{\mathrm{III}}(y) + f^{\mathrm{III}}(x)] +$$

$$+ \frac{1}{384}\int_{-\tau/2}^{+\tau/2} (\tau^2 - 4u^2)(5\tau^2 - 4u^2) f^{\mathrm{V}}(m+u)\, du.$$

The expansion may be easily continued in the indicated manner, but for the end I have in view that deduced above goes far enough. If according to this formula we replace $\dot{z}_2 - \dot{z}_1$ by $\frac{\tau_3}{2}(\ddot{z}_2 + \ddot{z}_1) - \frac{\tau_3^3}{24}(z_2^{\mathrm{IV}} + z_1^{\mathrm{IV}})$, terms of the fifth order are neglected, and if we replace $(3 z_2 \dot{z}_2 - 2 z_2^{\mathrm{III}}) - (3 z_1 \dot{z}_1 - 2 z_1^{\mathrm{III}})$ by

$$\frac{\tau_3}{2}(3\dot{z}_2^2 + 3z_2\ddot{z}_2 - 2z_2^{\mathrm{IV}} + 3\dot{z}_1^2 + 3z_1\ddot{z}_1 - 2z_1^{\mathrm{IV}}),$$

we neglect a quantity of the third order. I propose to terminate the expansion of $2\,\frac{\Delta P_2 Z P_1}{\sqrt{p}}$ with the term in τ_3^7, then the above mentioned substitutions will not alter the order of approximation. So we obtain the following approximate formula:

$$2\frac{\Delta P_2 Z P_1}{\sqrt{p}} = \frac{\tau_3}{1} - \frac{z_1 + z_2}{2}\frac{\tau_3^{\,3}}{3!} + \left\{\tfrac{1}{2}\dot{z}_2^{\,2} + \ddot{z}_2 + \tfrac{1}{2}\dot{z}_1^{\,2} + \ddot{z}_1\right\}\frac{\tau_3^{\,5}}{5!}$$

$$- \left\{(4z_2\ddot{z}_2 + 5\tfrac{1}{2}\dot{z}_2^{\,2} + 4\tfrac{1}{4}z_2^{\,\mathrm{IV}} + \tfrac{1}{2}z_2^{\,3}) + (4z_1\ddot{z}_1 + 5\tfrac{1}{2}\dot{z}_1^{\,2} + 4\tfrac{1}{4}z_1^{\,\mathrm{IV}} + \tfrac{1}{2}z_1^{\,3})\right\}\frac{\tau_3^{\,7}}{7!}$$

The development is symmetrical with respect to z_1 and z_2 and their derivatives, and resolves itself into two parts, which have the same form, and which depend besides on τ_3, only on the value of z and those of the derivatives of z at one point.

If the following series

$$\tau - z_1\frac{\tau^3}{3!} + (z_1^{\,2} + 2\ddot{z}_1)\frac{\tau^5}{5!} - (8z_1\ddot{z}_1 + 11\dot{z}_1^{\,2} + 8\tfrac{1}{2}z_1^{\,\mathrm{IV}} + z_1^{\,3})\frac{\tau^7}{7!}\cdot\cdot$$

where only the odd powers of the variable τ occur, be denoted by $U_1(\tau)$ and the corresponding series for z_2 and its derivatives by $U_2(\tau)$, we get:

$$2\frac{\Delta P_2 Z P_1}{\sqrt{p}} = \tfrac{1}{2}\{U_1(\tau_3) + U_2(\tau_3)\}$$

and the ratios of the triangles may be expressed in the following way in these functions U:

$$\frac{\Delta P_2 Z P_1}{\Delta P_3 Z P_1} = n_3 = \frac{U_1(\tau_3) + U_2(\tau_3)}{U_1(\tau_2) + U_3(\tau_2)} \quad . \; . \; . \; . \quad (VIIa)$$

and

$$\frac{\Delta P_3 Z P_2}{\Delta P_3 Z P_1} = n_1 = \frac{U_2(\tau_1) + U_3(\tau_1)}{U_1(\tau_2) + U_3(\tau_2)} \quad . \; . \; . \; . \quad (VIIb)$$

In the series $U(\tau)$ only such differential quotients occur as can be rationally expressed in p and $\frac{1}{a}$. By means of the known differential equations of the 1st and of the 2nd order for r

$$\dot{r}^2 = \frac{2}{r} - \frac{1}{a} - \frac{p}{r^2} \text{ and } \ddot{r} = \frac{p}{r^3} - \frac{1}{r^2}$$

we obtain by differentiating $z = \frac{1}{r^3}$

$$\dot{z}^2 = 9z^3\left(2 - \frac{r}{a} - \frac{p}{r}\right)$$

$$\ddot{z} = 3z^2\left(9 - 4\frac{r}{a} - 5\frac{p}{r}\right)$$

while from the differential equation $z^{\mathrm{III}} = 5\frac{\dot{z}\ddot{z}}{z} - \frac{40}{9}\frac{\dot{z}^3}{z^2} - z\dot{z}$ by differentiation with respect to τ and by elimination of z^{III} the following expression is found for z^{IV}.

$$z^{IV} = -z^3\left(6\frac{\dot{z}^2}{z^3} + \frac{\ddot{z}}{z^2} - 5\frac{\ddot{z}^2}{z^4} - \frac{20}{3}\frac{\ddot{z}\,\dot{z}^2}{z^2\,z^3} + \frac{40}{3}\frac{\dot{z}^4}{z^6}\right)$$

As $\frac{1}{a}$, p and the 3 heliocentric distances r are known, $\dot{z}^2$, $\ddot{z}$ and z^{IV} can be computed for each of the 3 points P_1, P_2 and P_3.

For a circular orbit all the derivatives of z are equal to zero and the function U becomes $\frac{\sin \tau \sqrt{z}}{\sqrt{z}}$. According to the preceding development we obtain for an elliptic orbit the following approximate formula for U, which still contains the 6th power of the interval:

$$U = \frac{\sin \tau \sqrt{z}}{\sqrt{z}} + \frac{\tau^5}{20} z^2 \left(9 - 4\frac{r}{a} - 5\frac{p}{r}\right). \quad . \quad . \quad (VIII)$$

By means of the values which take U_1, U_2 and U_3 for the values τ_1, τ_2 and τ_3 of the argument τ, we obtain for n_1 and n_3 values containing the terms of the 5th order with respect to the intervals; while the approximation may be extended to the 6th order, if we add to the above mentioned expression for U:

$$-\frac{\tau^7}{5040} z^3 (-\lambda - 120\mu + 170\lambda^2 + 340\lambda\mu - 1020\mu^2).$$

where λ and μ denote:

$$\lambda = \frac{3}{2}\left(9 - 4\frac{r}{a} - 5\frac{p}{r}\right)$$

$$\mu = 3\left(2 - \frac{r}{a} - \frac{p}{r}\right).$$

Astronomy. — "*Supplement to the account of the determination of the longitude of St. Denis (Island of Réunion), executed in* 1874, *containing also a general account of the observation of the transit of Venus*". By Prof. J. A. C. Oudemans.

When I set about to correct the imperfections left in my first communication, I began by calculating for the times of observation of the occultations the correction of Newcomb's parallactic correction, mentioned on p. 603 of my previous paper; as said there this correction amounts to

$$+ 0''67 \sin D + 0''05 \sin (D - g) - 0''09 \sin (D + g'),$$

where D stands for the mean elongation of the moon from the sun, g for the moon's mean anomaly, and g' for that of the sun.