Mathematics. — "*On the number of common tangents of a curve and a surface.*" By Dr. W. A. VERSLUYS. (Communicated by Prof. D. J. KORTEWEG).

§ 1. Let C_1 be a plane algebraic curve of class r_1 and S_2 an algebraic surface of class m_2. Every tangent of C_1 touching S_2 is a tangent of the section s of the surface S_2 with the plane V of C_1. Conversely, each common tangent of C_1 and s is a common tangent of C_1 and S_2. The curves C_1 and s being of the class r_1 and m_2 respectively, they have $r_1 m_2$ common tangents. Hence, S_2 and C_1 have $r_1 m_2$ common tangents too.

Let the plane V of C_1 occupy the particular position of touching S_2 in δ points of ordinary contact and in χ points of stationary contact, the class of the section s is now

$$m_2 - 2\delta - 3\chi\ ^{1)}.$$

Hence, the curves s and C_1 have now

$$r_1 (m_2 - 2\delta - 3\chi)$$

common tangents. Every tangent of C_1 passing through one of the points of contact δ and χ is a common tangent of C_1 and S_2, without being a common tangent of C_1 and s. In § 8 will be proved, that, if D_1 be the developable formed by the tangents of C_1, each generating line of D_1 touching S_2 in a point δ counts for two common tangents of C_1 and S_2 and each generator of D_1 touching S_2 in a point χ counts for three common tangents of C_1 and S_2. If C_1 be a plane curve, the developable D_1 is the plane V counted r_1 times. Every ordinary contact δ gives thus $2r_1$ common tangents of C_1 and S_2, and every stationary contact χ gives $3r_1$ common tangents of C_1 and S_2. Thus, the total number of common tangents is

$$r_1 (m_2 - 2\,\delta - 3\,\chi) + 2\,\delta\, r_1 + 3\,\chi\, r_1 = r_1\, m_2.$$

If the plane V of C_1 and the surface S_2 touch along a line, then every tangent of C_1 touches S_2 and the number of common tangents becomes infinite. This case presents itself if S_2 be a developable and V one of its tangent planes. Every tangent of C_1 touches S_2 twice, if S_2 be a torus and V be one of the planes touching S_2 along a circle.

If C_1 be a curve in space the number of common tangents of C_1 and S_2 is still $r_1 m_2$, where r_1 is the rank of C_1. This will be proved first for some special curves and surfaces and afterwards for the general case.

§ 2. Let S_2 be a cone with vertex T; the projection of a common

1) VERSLUYS, These Proceedings. May 27, 1905.

tangent of C_1 and S_2 on an arbitrary plane V, not passing through the centre of projection T, is a common tangent of the projection p_1 of C_1 and of the section s of S_2 with V. The converse is equally true. The class of p_1 and s being r_1 and m_2 respectively, the number of common tangents of p_1 and s and thus of C_1 and S_2 is $r_1 m_2$.

If S_2 be a developable D_2, a tangent t of C_1 touches D_2 if a tangent plane of D_2 pass through t, and conversely. Let D'_1 and C''_2 be the polar reciprocals of C_1 and D_2. To a plane of D_2 passing through a tangent t of C_1 corresponds a point of C'_2 on a generator t' of D'_1, and conversely. The number of these intersections of C'_2 and D'_1 is $r_1 m_2$, for the curve C''_2 is of order m_2 and the developable D'_1 of order r_1. Then, since there are $r_1 m_2$ planes of D_2 passing through tangents of C_1, these are $r_1 m_2$ common tangents of C_1 and S_2.

We can show in a very simple way, that if the curve C_1 be an arbitrary algebraic curve of rank r_1, and the surface S_2 be an arbitrary algebraic surface of class m_2, the number of common tangents is still $r_1 m_2$. For the tangents to S_2 form a complex of order m_2, and the tangents of C_1 form a ruled surface of order r_1. Now according to a theorem due to HALPHEN [1]) the number of their common rays is $r_1 m_2$, which proves the proposition.

§ 3. Some theorems concerning the contact of a developable with an arbitrary surface will be deduced from the theorem proved above.

Let C_1 be a twisted cubic C^3 and D^4 the developable formed by its tangents. Let S_2 be an arbitrary surface of order n_2, having a cuspidal and a nodal curve respectively of order $\mathfrak{r}_2$ and ξ_2 and let D^4 and S_2 have an ordinary contact in σ and a stationary contact in χ points, whilst none of the tangents of C^3 is an inflexional tangent of S_2 and C^3 does not touch S_2. The number of common tangents of C^3 and S_2 is now also for this particular position $r_1 m_2$ or $4 m_2$. These common tangents of C^3 and S_2 are: 1st the tangents of C^3 touching the curve of intersection s of D^4 and S_2 and 2nd the tangents of C^3 touching S_2 in the points σ and χ where the surfaces D^4 and S_2 touch. Let every common tangent of C^3 and S_2 passing through an ordinary point of contact σ count for x common tangents and let every common tangent through a stationary point of contact χ count y times.

The number of common tangents of C^3 and s will be

$$4 m_2 - x \sigma - y \chi.$$

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[1]) R. STURM, Linien Geometrie, I p. 44.

Let K be the developable formed by the tangents of s. Let l be a common tangent of C^3 and S_2, touching C^3 in R and s in P. One of the tangents of C^3 consecutive to l meets S_2 in two real points of s, consecutive to P, as l is supposed to be no principal tangent of S_2. The osculating plane V of C^3 in R contains therefore four consecutive points of s, so it is a stationary plane α of s in P. Consequently the plane V is also a stationary tangent plane of K along the generating line l. So C^3 has in R three consecutive points in common with K and no more.

The $3n_2$ points where C^3 meets S_2 are cusps β of s [1]), so they are triple points on the developable K.

Each of these $3n_2$ points β counts at least for three points of intersection of C^3 with K. Each of these points β counts for not more than three points of intersection, as we have assumed that C^3 does not touch S_2, and the tangent in β to C^3 does not lie in the triple tangent plane of K in β, which triple tangent plane coincides with the osculating plane of s in β, i. e. with the tangent plane of S_2 in β.

The curve C^3 meets K only in the $4m_2 - x\delta - yx$ points R and in the $3n_2$ points β, as every tangent to s lies in an osculating plane of C^3, and through a point of C^3 no plane can pass osculating C^3 still elsewhere. The order of K or the rank of s is

$$r = 4m_2 + 3n_2 - 2\delta - 3\chi \text{ }^{2)}.$$

So the number of points of intersection of C^3 and K is

$$3(4m_2 + 3n_2 - 2\delta - 3\chi).$$

As the only points of intersection of C^3 and K are the points R and β counted three times, we find the relation

$$3(4m_2 + 3n_2 - 2\delta - 3\chi) = 3 \times 3n_2 + 3(4m_2 - x\delta - y\chi)$$

from which ensues

$$x = 2, \qquad y = 3,$$

or in words:

If the developable D^4 and an arbitrary surface S_2 have an ordinary contact, two consecutive generating lines of D^4 touch S_2.

If the developable D^4 and an ordinary surface S_2 have a stationary contact, three consecutive generating lines of D^4 touch S_2.

These theorems hold good too in the case that the developable is a cone [3]).

1) VERSLUYS, Mém. de Liège. 3me série, T. VI, 1905. Sur les nombres Plückériens etc.

2) VERSLUYS, These Proceedings May 27, 1905.

3) VERSLUYS, These Proceedings May 27, 1905.

The two theorems mentioned above and their reciprocals and some special cases will now be treated algebraically.

§ 4: Let C_1 be a rational twisted curve of rank r_1 and S a surface of order n, possessing no multiple curves. Let

$$ax + by + cz + d = 0$$

represent the osculating plane of C_1, a, b, c, d being integer rational algebraic functions of t. Differentiating we find for an arbitrary tangent of C_1 equations of the form

$$a_1 x + b_1 y + c_1 z + d_1 = 0,$$
$$a_2 z + b_2 y + c_2 z + d_2 = 0.$$

Solving y and z in function of x and t we find:

$$y = \frac{Ax + B}{C}, z = \frac{Dx + E}{C}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

in which A, B, C, D and E are functions in t of order r_1. If we substitute the values (A) in the equation of the surface S, we arrive at an equation (B), which is in x of order n and in t of order $n\,r_1$. For every value of t this equation (B) furnishes the n values of x belonging to the points of intersection of a tangent l to C_1. If two of these values become equal, the tangent l will meet the surface S in two consecutive points and as S is supposed to have no multiple curves the tangent l will also be a tangent of S. Those tangents of C_1 are excluded which are at right angles with the X-axis, all points of intersection with S possessing the same x; so all roots x coincide, without the points of intersection coinciding. Every line being at right angles with the X-axis meets the line at infinity in the plane $x = 0$. So the number of these particular tangents of C_1 is r_1.

The equation (B) has two equal roots in x for a certain value of t, when this value of t causes the discriminant of (B) to vanish. The discriminant is in the coefficients of (B) of order $2\,(n-1)$ and as the coefficients of (B) are of order $r_1\,n$ in t, the discriminant is of order $2\,r_1\,n\,(n-1)$ in t.

By a parallel displacement of the axes the plane $x = 0$ can be made to pass through one of the tangents of C_1 which is at right angles with the X-axis.

Writing $t + q$ for t, we can take q in such a way that this tangent of C_1 lying in $v = 0$ corresponds to the value $t = 0$. The equation (B) has then passed into an equation (B') where for $t = 0$ all roots x vanish.

The first equation (A)

$$y = \frac{A\,x + B}{C} \quad or \quad x = \frac{C\,y - B}{A}$$

must now pass into $x = 0$ for $t = 0$, so that C and B must contain, after the change of variables, t as a factor, A not being divisible by t. As the projection on the plane $x = 0$ of the tangent lying in this plane can be any arbitrary line and as C vanishes for $t = 0$, D and E must also vanish for $t = 0$. In the equation (B') the coefficient of x^n will be divisible by t and the coefficient of x^i divisible by t^{n-i}.

According to SALMON [1]) the discriminant of equation (B') will be divisible by $t^{n(n-1)}$. For each one of the r_1 particular tangents of C_1 which are at right angles with the X-axis $n(n-1)$ roots of the discriminant of equation (B) become equal. That discriminant possessing $2r_1\,n(n-1)$ roots, there are left $r_1\,n(n-1)$ roots, to each of which corresponds an equation (B), possessing two equal roots. So there are $r_1\,n(n-1)$ tangents of C_1 which also touch S. As S possesses no multiple curves the class m is $n(n-1)$. The number of common tangents of C_1 and S is thus as before mentioned

$$r_1\,m.$$

§ 5. So far we have supposed that C_1 occupies no particular position with respect to S. For particular positions of C_1 two or more of the common tangents of C_1 and S can become consecutive tangents of C_1. Let t be a tangent of C_1 touching S in P, and let a tangent of C_1 consecutive to t be also a tangent of S. The developable D_1 formed by the tangents of C_1 and the surface S will touch in P. We shall now investigate when the contact is ordinary and when stationary.

For simplification I assume for C_1 the twisted cubic C^3

$$(x = p + t,\ y = t^2,\ z = t^3).$$

The equation of the developable D_1 or D^4 is now:

$$z^2 - 6\,(x + p)\,y\,z + 4\,y^3 + 4\,(x + p)^3\,z - 3\,(x + p)^2\,y^2 = 0,$$

or

$$0 = z - \frac{3}{4\,p}\,y^2 + etc. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

If we choose for point P where the surface S touches D^4 the origin of the coordinates the equation of S is

$$0 = z + ax^2 + 2\,h\,xy + by^2 + etc. \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

The surfaces D^4 and S have stationary contact in the origin when

[1]) Modern Higher Algebra, § 111, note.

$$a\left(b+\frac{3}{4p}\right)-h^2=0.\text{ [1]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C)$$

The equation of an osculating plane of C^3 is

$$t^3-3(x+p)t^2+3yt-z=0.$$

The equations of a tangent to C^3 are

$$t^2-2(x+p)t+y=0,\quad (x+p)t^2-2yt+z=0,$$

or

$$y=2(x+p)t-t^2,\quad z=3(x+p)t^2-2t^3.$$

Substitution these values of y and z in the equation of S, we find an equation of order n in x

$$0=a_0+a_1x+a_2x^2+a_3x^3+\text{etc.}$$

where

$$a_0=3pt^2+4bp^2t^2-2t^3-4bpt^3+\text{etc.},$$
$$a_1=4hpt+3t^2-2ht^2+8bpt^2-4bt^3+\text{etc.},\quad . \quad (D)$$
$$a_2=a+4ht+4bt^2+\text{etc.}$$

The discriminant of this equation is of the form

$$a_0\varphi+a_1{}^2\psi^2,\text{ [2]}.$$

As a_0 and a_1 contain respectively t^2 and t as a factor, whilst φ and ψ are in general not divisible by t, the discriminant is divisible by t^2 or the discriminant has two roots $t=0$. As to every root of the discriminant (except the particular $n(n-1)$-fold ones) corresponds a common tangent of C^3 and S, the X-axis counts here for two common tangents of C^3 and S, or the two consecutive tangents of C^3 lying in the common tangential plane of D and S both touch also S.

The discriminant is a determinant, which gives when developed according to the elements of the first two columns

$$\{2na_0a_2-(n-1)a_1{}^2\}\varphi_1+n^2a_0{}^2\varphi_2+na_0a_1\varphi_3+a_1{}^2\varphi_4 \quad . \quad (E)$$

§ 6. If the X-axis does not coincide with one of the inflexional (or principal) tangents of S in the origin P, then the Y-axis can be taken so that $h=0$; to this end we have but to take for Y-axis the diameter of the indicatrix conjugate to the X-axis. The expressions for the coordinates of a point on C^3 will not change if we now also take for plane $x=0$ the plane determined by the new Y-axis, and one of the two tangents of C^3 meeting the Y-axis outside P, and for plane at infinity the osculating plane of C^3 in the point where C^3 touches the new plane $x=0$, whilst for plane $y=0$ is taken the

1) Salmon, Three Dim. § 204.

2) Salmon, Modern Higher Algebra. § 111.

plane determined by the X-axis and the point where C^3 touches $x = 0$. When $h = 0$ the terms of the lowest order in t in the coefficients a_0, a_1 and a_2 are respectively of order 2, 2 and 0.

The terms of the lowest order in t of the discriminant appear in the first term of the equation (E) at the end of the preceding §, namely in the term $2na_0a_1\varphi_1$. So the terms of the lowest order in t are

$$Ca\,(3p + 4bp^2)\,t^2$$

where C represents a constant. The discriminant possesses three roots $t = 0$ or the X-axis counts for three common tangents of C^3 and S, if

$$a\,(3p + 4bp^2) = 0$$

or if

$$a = 0, \quad 3 + 4bp = 0, \quad p = 0.$$

If $9 + 4bp = 0$, the surfaces D^4 and S have according to (C) a stationary contact, as h is also equal to nought. The origin P is now an ordinary point (not a parabolic or double point) on the surface S and the common tangent (the X-axis) does not coincide with one of the inflexional tangents of S in P.

This furnishes the theorem:

If an arbitrary surface S and a developable D^4 have a stationary contact in an ordinary point P of both surfaces and the generating line l of D^4 through P is neither of the two inflexional tangents of S in P, then l counts for three common tangents of the cuspidal curve C^3 and of S.

If $a = 0$ the surfaces D^4 and S have according to (C) still a stationary contact, as still $h = 0$. The origin P is now a parabolic point of S whilst the X-axis is the only inflexional tangent. The coefficients a_0, a_1 and a_3 all contain the factor t^2. So the discriminant possesses the factor t^4, so that now the discriminant has four roots $t = 0$. So the X-axis now counts for four common tangents of C^3 and S.

If $p = 0$, then C^3 touches S in the origin P, whilst the osculating plane of C^3 in P coincides with the tangent plane of S in P. The terms of the lowest order in t in the coefficients a_0, a_1 and a_2 are now respectively of order 3, 2 and 0. So the discriminant (E) is divisible by t^3, so that C^3 and S now have in the origin P three common tangents. Writing in the equation (B) of the surface S for the coordinates of a point on C^3 the expressions $x = t, y = t^2, z = t^3$, we obtain an equation in t, containing t^2 as a factor. The curve C^3 has thus in the origin only two points, but three tangents in common with S.

If $h = a = p = 0$, then C^3 touches the surface S in a parabolic point P, the tangent in P to C^3 coincides with the principal tangent in P of S, whilst the osculating plane of C^3 in P coincides with the tangent plane of S in P. From the expressions (D) for a_0, a_1 and a_0 follows that the discriminant (E) is divisible by t^4, so that C^3 and S have now four common tangents in common in the point P.

If $h = b = p = 0$ then C^3 touches S still in a parabolic point; the only difference to the preceding case is that C^3 no longer touches the principal tangent. From the equations (D) and (E) ensues that C^3 and S possess only three common tangents.

If $h = b = 0$ and $p \gtrless 0$, then P is a parabolic point for which the principal tangent does not coincide with the tangent to C^3. From (D) and (E) ensues now readily that the X-axis counts but for two common tangents of C^3 and S.

§ 7. When the X-axis coincides with one of the principal tangents of S in P then the axes cannot be taken in such a way that $h = 0$; but we have now $a = 0$. The terms of the lowest order in t in the coefficients a_0, a_1, a_2 (D) are now respectively of degree 2, 1. 1. So the discriminant (E) is only divisible by t^2, so that now the X-axis counts for two common tangents of C^3 and S. The X-axis itself has now with S in P three consecutive points in common, so it counts already for two common tangents. A tangent of C^3 following the X-axis does not touch S any more.

The term of the second degree in t of the discriminant (E) has now for coefficient $16\, C h^2 p^2$, where C is a constant. So the discriminant has three roots $t = 0$, when $h = 0$ or $p = 0$. The case $h = 0$ is just the one treated in § 6.

If $p = a = 0$ then C^3 touches in P one of the principal tangents of S in P, whilst the osculating plane of C^3 in P still coincides with the tangent plane of S in P. Out of the expressions (D) for a_0, a_1 and a_2 it is evident that these coefficients are respectively divisible by t^3, t^2 and t. So the discriminant (E) is divisible by t^4 or it has four roots $t = 0$. The X-axis counts thus for four common tangents of C^3 and S. By substitution of $x = t$, $y = t^2$, $z = t^3$ in the equation (B) of the surface S we find that C^3 and S now have in the origin three consecutive points in common.

§ 8. Let C_1 now be an arbitrary twisted curve and D_1 the developable formed by its tangents and let D_1 touch the arbitrary surface S in P. Let l be the generating line of D_1 touching S in P and let R be the point, in which it touches C_1. Let V be the osculating plane of C_1 in R. Through R and five points of C_1 consecutive to R a twisted cubic C^3 can be brought, on the condition that R, l and V are an ordinary point, an ordinary tangent and an ordinary osculating plane of C_1. The developable D^1 formed by the tangents to C^3 and the developable D_1 have in common the line l and four consecutive generating lines.

If l must count for 2, 3 or 4 common tangents of C^3 and S, this is also the case for C_1 and S. The theorems proved in § 6 and 7 for C^3 hold good for any twisted curve. This gives rise to the following theorems:

If the developable D_1 corresponding to curve C_1 touches any surface S in point P whilst the generating line l of D_1 through P is no inflexional tangent of S, the line l counts for two or for three common tangents to C_1 and S according to the surfaces having in P an ordinary or a stationary contact.

If the point of contact P of D_1 and S be a parabolic point on S, then l counts for four or for two common tangents of C_1 and S according as the inflexional tangent of S in P coinciding with l or not.

If the point of contact P of D_1 and S be a hyperbolic point on S and if the tangent l of C_1 coincides with an inflexional tangent in the point P of S, then l counts for four or for two common tangents of C_1 and O according to R coinciding with P or not.

If C_1 touches S in P, whilst the osculating plane of C_1 in P coincides with the tangent plane of S in P, then the tangent l in P to C_1 counts for four or for three common tangents of C_1 and O, according to l being an inflexional tangent of O in P or not.

The theorems proved here for curves in space hold with a slight modification (see § 1) still for plane curves. They can be easily proved by taking for C_1 first a parabola p^2 after which they can be directly extended to an arbitrary conic section and after this to an arbitrary plane curve.

Delft, June 1905.

Physics. — *The shape of the sections of the surface of saturation normal to the x-axis, in case of a three phase pressure between two temperatures."* By Prof. J. D. VAN DER WAALS.

In these Proceedings of March 1905 I have (fig. 4, 5 and 6) represented in a diagram some sections of the (p, T, x)-surface normal to the T-axis for three temperatures, at which three phases can exist simultaneously. The three temperatures chosen were: 1st the