Citation:

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not disappear on shaking for a while, but if the liquid was allowed to stand over night it became homogeneous, and on distillation in vacuo yielded diisobutylformamide.

Formic esters of unsaturated glycols also seem to react readily with amines, at least Mr. W. VAN DORSSEN, who is engaged in the Utrecht laboratory upon the study of the 3.4-dihydroxy-1.5-hexadiene

$$\begin{array}{l} CH_2 = CH - CH.OH \\ \qquad\qquad\quad | \\ CH_2 = CH - CH.OH \end{array}$$

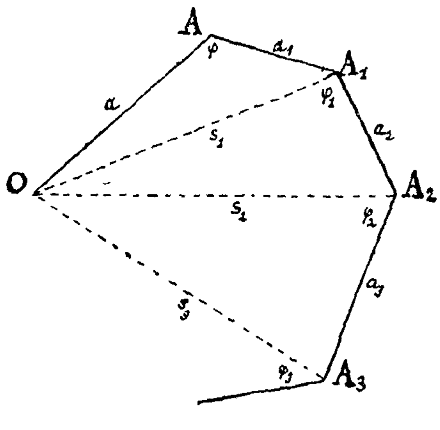
obtained, on mixing 1 gram of the diformate of this glycol with 1.3 gram of benzylamine, a rise in temperature from 18° to 65°, and after distilling off the glycol could readily isolate benzylformamide m. p. 61°.

Mathematics. — *"A local probability problem"*. By Prof. J. C. KLUYVER.

The following problem was lately (Nature, July 27) proposed by Prof. PEARSON:

"A man starts from a point O, and walks l yards in a straight line; he then turns through any angle whatever, and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r + dr$ from his starting point O." [1])

I find that the general solution of this problem depends upon the theory of BESSEL's functions, especially that in some particular cases it leads to the evaluation of certain definite integrals, involving these functions.



Let $OAA_1A_2A_3 \ldots A_{n-1}$ be the broken line, the n stretches of which need not be all of the same length. Then the shape of the figure, not its orientation in the plane, is wholly determined by the lengths $a, a_1, a_2, \ldots, a_{n-1}$ of the stretches, and by the magnitudes of the angles $\varphi, \varphi_1, \ldots, \varphi_{n-2}$, formed at the origin of each stretch a_k by the stretch itself and by the radius vector s_{k-1}.

1) Recently (Nature, August 10) Prof. PEARSON stated, that the solution for n very large was already virtually contained in a memoir on sound by Lord RAYLEIGH.

In a turning point the rambler takes his new direction at random; hence for any angle φ_k, all values between 0 and 2π have an equal chance, and the probability that those angles are respectively included within the intervals, $\varphi_k, \varphi_k + d\varphi_k$, is equal to the product

$$\frac{1}{(2\pi)^{n-1}} d\varphi\, d\varphi_1 \ldots d\varphi_{n-2}.$$

If we integrate this product over a region, determined by the condition that the n^{th} radius vector s_{n-1} remains less than a given distance c, the result will be the required probability $W_n(c; aa_1a_2 \ldots a_{n-1})$, that the ending point of the path lies within the distance c from the starting point O.[1])

The integration becomes less complicated, if we introduce in the usual way a discontinuous factor. Choosing a function $T(\varphi, \varphi_1, ., \varphi_{n-2})$ such, that it vanishes when $s_{n-1} > c$, and that it is equal to unity for $s_{n-1} < c$, to each of the variables φ_k we may give the whole range from 0 to 2π, and we have

$$W_n(c; aa_1 \ldots a_{n-1}) = \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi}\int_0^{2\pi} \ldots \int_0^{2\pi} d\varphi d\varphi_1 \ldots d\varphi_{n-2}\, T(\varphi, \varphi_1, \ldots, \varphi_{n-2}).$$

For the function T we may take WEBER's discontinuous integral, that is, we may put

$$T(\varphi, \varphi_1, ., \varphi_{n-2}) = c \int_0^{\infty} J_1(uc) J_0(us_{n-1}) du,$$

the integral being equal to zero or to unity according to s_{n-1} being larger or smaller than c.

This choice of the factor T makes a good deal of reduction possible.

If we consider the side c of a triangle as a function of the sides a and b and of the inclosed angle C, the relation holds

$$J_0(ua) J_0(ub) = \frac{1}{2\pi} \int_0^{2\pi} J_0(uc)\, dC,$$

and this formula can be repeatedly used in reducing the integral $W_n(c; aa_1 \ldots a_{n-1})$.

So we get successively

1) In the case $n = 2$, we have, supposing $a + a_1 > c > a - a_1$, $W_2(c; aa_1) = \frac{1}{\pi} \arccos \frac{a^2 + a_1^2 - c^2}{2aa_1}$. Of course for $c > a + a_1$ W_2 becomes equal to unity and it is zero for $a - a_1 > c$.

$$J_0(us_{n-2})\, J_0(ua_{n-1}) = \frac{1}{2\pi}\int_0^{2\pi} J_0(us_{n-1})\, d\varphi_{n-2}\,,$$

$$J_0(us_{n-3})\, J_0(ua_{n-2}) = \frac{1}{2\pi}\int_0^{2\pi} J_0(us_{n-2})\, d\varphi_{n-1}\,,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$J_0(ua)\, J_0(ua_1) = \frac{1}{2\pi}\int_0^{2\pi} J_0(us_1)\, d\varphi\,,$$

and consequently

$$W_n(c\,;\, aa_1 \ldots a_{n-1}) = c\int_0^{\infty} J_1(uc)\, J_0(ua)\, J_0(ua_1) \ldots J_0(ua_{n-1})\, du.$$

From this result we infer, that the probability sought for is of a rather intricate character. The $n+1$ functions J are oscillating functions, and have their signs altering in an irregular manner as the variable u increases. Hence even an approximation of the integral is not easily found, and as a solution of PEARSON's problem it is little apt to meet the requirements of the proposer.

From a mathematical point of view the integral presents some interest. In fact, if we consider it as a function of c, it is readily seen to be continuous and finite for all real values of c, and the same holds for a certain number of derivatives with respect to c, but a closer inspection shows, that this analytic expression, regularly built up as it is, represents in different intervals different analytic functions. To make good this assertion, we have only to remember that the integral stands for the probability required in PEARSON's problem. Hence we know beforehand, that it always must be positive and increasing with c, but that it never surpasses 1, this upper limit being actually reached as soon as c becomes greater than $a+a_1+\ldots+a_{n-1}$. Moreover, if we suppose $a > a_1 + a_2 + \ldots + a_{n-1}$, the inequality $a > c + a_1 + a_2 \ldots + a_{n-1}$ is possible for small values of c. And if the latter inequality holds, the rambler of PEARSON's problem necessarily arrives outside the circle with radius c, and the probability is zero.

Thus, by solving the problem, we have found

$$\left.\begin{matrix} c > a + a_1 \ldots + a_{n-1}\,, \ldots 1 \\ a > c + a_1 \ldots + a_{n-1}\,, \ldots 0 \end{matrix}\right\} = c\int_0^{\infty} J_1(uc)\, J_0(ua)\, J_0(ua_1) \ldots J_0(ua_{n-1})\, du\,,$$

quite independently of the number of the J_0-functions, showing thereby at the same time, that the continuous analytic expression cannot be regarded as a single analytic function.

The same still holds for values of c, not fulfilling one of the above inequalities, though the integral is then continuously varying with c.

So for instance in the case $n = 3$, taking the stretches $a > a_1 > a_2$ in such a manner, that a triangle is possible having these sides, I am led to conclude from the discontinuities of the first derivative that in each of the following intervals

$$\begin{array}{llll} I & a_1 + a_2 - a > c > 0 & IV & a + a_1 + a_2 > c > a + a_1 - a_2 \\ II & a - a_1 + a_2 > c > a_1 + a_2 - a & V & c > a + a_1 + a_2 \\ III & a + a_1 - a_2 > c > a - a_1 + a_2 & & \end{array}$$

a distinct analytic function is defined by the integral.

Some further remarks may be made. On integrating by parts we find

$$W_n\,(c;\, aa_1 \ldots a_{n-1}) = 1 - a \int_0^\infty J_1\,(ua)\, J_0\,(uc)\, J_0\,(ua_1) \ldots J_0\,(ua_{n-1})\, du$$

$$- a_1 \int_0^\infty J_1\,(ua_1)\, J_0\,(ua)\, J_0\,(uc) \ldots J_0\,(ua_{n-1})\, du$$

$$\cdots\cdots\cdots\cdots\cdots$$

or what is the same:

$$1 = W_n\,(c;\, aa_1 \ldots a_{n-1}) + W_n\,(a;\, ca_1 \ldots a_{n-1}) + W_n\,(a_1;\, ac \ldots a_{n-1}) + \ldots\ldots$$

Dividing both sides of the equation by $n + 1$ we may interpret the coming relation as follows: $n + 1$ equal or unequal stretches being given, if n of them, taken at random, are put together to a broken line, according to the rules of PEARSON's problem, the probability is equal to $\frac{1}{n+1}$, that the distance between the extremities of this broken line is less than the stretch that was left out.

And from the same equation we deduce in the very particular case $c = a = a_1 \ldots = a_{n-1}$

$$W_n\,(a\,;\, a^n) = \frac{1}{n+1}\,,$$

or: the rambler of PEARSON's problem after walking along n equal stretches has the chance $\frac{1}{n+1}$ to find himself within a stretches' length from his starting point.

In the most general case of the problem I cannot give a practical solution; something however can be done, in the case: n very large, all stretches equal, treated already by Lord RAYLEIGH.

Putting $na = L$, $c = \frac{L}{a}$, we have

$$W_n(c; a^n) = W_n(c/L) = \int_0^\infty J_1(u)\, J_0\left(\frac{au}{n}\right)^n du.$$

Now by raising to the n^{th} power the ordinary power series for $J_0\left(\frac{au}{n}\right)$ we get

$$J_0\left(\frac{au}{n}\right)^n = 1 + \sum_{k=1}^{k=\infty} \frac{(-1)^k}{k!} \cdot \left(\frac{au}{2}\right)^{2k} \cdot \frac{S_k(n)}{k!\, n^{2k}},$$

where $S_k(n)$ stands for the sum of squares of the coefficients of the expansion $(u_1 + u_2 + \ldots u_n)^k$, so that

$$\frac{S_1(n)}{1!\, n^2} = \frac{1}{n}, \quad \frac{S_2(n)}{2!\, n^4} = \frac{1}{n^2} - \frac{1}{2n^3}, \quad \frac{S_3(n)}{3!\, n^6} = \frac{1}{n^3} - \frac{3}{2n^4} + \frac{2}{3n^5}.$$

Generally supposing n very large we may put approximately

$$\frac{S_k(n)}{k!\, n^{2k}} = \frac{1}{n^k},$$

and, substituting, we find that this approximation leads to the supposition

$$J_0\left(\frac{au}{n}\right)^n = e^{-\frac{a^2 u^2}{4n}}.$$

For small values of u the approximation is good enough. It is true both functions behave quite differently when u becomes very large, but as they are rather rapidly converging to zero, the actual amount of their difference can be neglected. In particular I find that the integral

$$\int_n^\infty J_1(u)\, J_0\left(\frac{au}{n}\right)^n du$$

is of an order of smallness certainly higher than that of the expression

$$\frac{2n}{n-2} \cdot \left(\frac{2}{\pi}\right)^{\frac{n+1}{2}} \cdot \left(\frac{1}{a}\right)^{\frac{n}{2}},$$

while the order of smallness of the integral

$$\int_n^\infty J_1(u)\, e^{-\frac{a^2 u^2}{4n}} du$$

is that of the expression

$$e^{-\frac{1}{4}a^2 n}\left(J_0(n) - J_0(q)\right), \quad q > n.$$

Hence if only a be rather greater than unity, both integrals cannot have an appreciable difference and we may put

$$W_n(c/L) = \int_0^\infty J_1(u) e^{-\frac{u^2 \alpha^2}{4n}} du = 1 - e^{-\frac{n}{\alpha^2}} = 1 - e^{-n\frac{c^2}{L^2}}.$$

From this result it is evident, that $W_n(c/L)$ for n very large is always nearly unity. The rambler, walking along a very great number of very short stretches will almost certainly arrive in the neighbourhood of his starting point.

Putting $c = \frac{1}{n} L$, we find $W_n(c/L) = 1 - e^{-\frac{1}{n}} = \frac{1}{n} - \frac{1}{2n^2} \ldots$, a result nearly equal to the true value $\frac{1}{n+1}$.

Returning to the general expression for $W_n(c; aa_1 \ldots a_{n-1})$ we observe the possibility of differentiating the integral with respect to c in the usual way a number of $2m$ times, provided $2m < \frac{n+1}{2}$.

Supposing $c > a + a_1 + \ldots + a_{n-1}$ and putting

$$J_0(ua) J_0(ua_1) \ldots J_0(ua_{n-1}) = f(u),$$

we deduce by differentiation

$$1 = c \int_0^\infty J_1(cu) f(u) du,$$

$$0 = \int_0^\infty u J_0(cu) f(u) du \quad , \quad 0 = \int_0^\infty u^2 J_1(cu) f(u) du,$$

$$0 = \int_0^\infty u^3 J_0(cu) f(u) du \quad , \quad 0 = \int_0^\infty u^4 J_1(cu) f(u) du,$$

$$\ldots\ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots$$

$$0 = \int_0^\infty u^{2m-1} J_0(cu) f(u) du \quad , \quad 0 = \int_0^\infty u^{2m} J_1(cu) f(u) du.$$

These equations allow us to introduce into the integral a new Bessel function, the function $J_{2m+1}(u)$. For $J_{2m+1}(u)$ is connected with $J_0(u)$ and $J_1(u)$ by the relation

$$J_{2m+1}(u) = P_{0,2m}(u) J_1(u) - P_{1,2m-1}(u) J_0(u),$$

where

$$P_{0,2m}(u) = \frac{1}{u^{2m}}(b_0 + b_2 u^2 + \ldots + b_{2m} u^{2m})$$

and

$$P_{1,2m-1}(u) = \frac{1}{u^{2m}}(b_1 u + b_3 u^3 + \ldots + b_{2m-1} u^{2m-1})$$

are a pair of SCHLÄFLI's polynomials.

Using this relation we obtain

$$b_0 = c^{2m+1} \int_0^\infty u^{2m} J_{2m+1}(uc) f(u)\, du,$$

and as

$$b_0 = \underset{u=0}{Lim}\, u^{2m} P_{0,2m}(u) = 2^{2m}\, m\,!$$

we have

$$2^{2m}\, m\,! = c^{2m+1} \int_0^\infty u^{2m} J_{2m+1}(uc)\, J_0(ua)\, J_0(ua_1) \ldots J_0(ua_{n-1})\, du$$

with the conditions

$$c > a + a_1 + \ldots + a_{n-1} \quad , \quad m < \frac{n+1}{4}.$$

Evidently the value of the integral would be zero, if instead of the first of these conditions the condition

$$a > c + a_1 + \ldots + a_{n-1}$$

was satisfied.

In the same manner we might differentiate and also integrate with respect to one or to several of the parameters a. This leads for instance to the following results

$$n \text{ even}: \quad 0 = \int_0^\infty J_1(uc)\, J_1(ua)\, J_1(ua_1) \ldots J_1(ua_{n-1})\, du$$

$$n \text{ odd}: \quad 0 = \int_0^\infty u\, J_1(uc)\, J_1(ua)\, J_1(ua_1) \ldots J_1(ua_{n-1})\, du.$$

$$c > a + a_1 + a_2 + \ldots + a_{n-1}.$$

Still other results present themselves when PEARSON's problem is slightly modified. Again putting

$$J_0(ua)\, J_0(ua_1) \ldots J_0(ua_{n-1}) = f(u)$$

and writing ϱ for c, we get by differentiation with respect to ϱ

$$W_n(d\Omega) = \frac{1}{2\pi}\,\varrho\, d\varrho\, d\theta \int_0^\infty u\, J_0(u\varrho)\, f(u)\, du,$$

and here $W_n(d\Omega)$ means the probability that the ending point of the broken line falls on a given element $d\Omega$ of the plane, the polar coordinates of which are ϱ, θ.

By integrating over a given finite region we may deduce the probability that the rambler reaches that region [1]).

First let the region be a rectangle R, and let the rectangular coordinates of its vertices be $\pm p$, $\pm q$, then we find for the corresponding probability

$$W_n(R) = \frac{1}{2\pi}\int_0^\infty u\, f(u)\, du \int_{-p}^{+p} d\xi \int_{-q}^{+q} d\eta\, J_0(u\sqrt{\xi^2+\eta^2}).$$

Now we have

$$J_0(u\sqrt{\xi^2+\eta^2}) = \frac{1}{2\pi}\int_0^{2\pi} \cos(u\,\xi\cos a)\cos(u\,\eta\sin a)\, da,$$

and therefore, effectuating the integrations with respect to ξ and to η,

$$W_n(R) = \frac{4}{\pi^2}\int_0^\infty u\, f(u)\, du \int_0^{\frac{\pi}{2}} \frac{\sin(pu\cos a)\sin(qu\sin a)}{u^2\sin a\cos a}\, da.$$

A somewhat simpler expression is found, if changing the variables we pass from u and a to

$$v = u\cos a,$$
$$w = u\sin a.$$

Then the probability $W_n(R)$ is expressed as follows:

$$W_n(R) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty dv\, dw\, \frac{\sin pv}{v}\cdot\frac{\sin qw}{w}\cdot f(\sqrt{v^2+w^2}).$$

Again an evaluation of this double integral is generally not practicable, but the problem itself gives the value of the integral, if both

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1) If this region is a circle with radius c, the centre of which lies at a distance b from the starting point O, we have at once

$$W_{n+1}(c;\, baa_1\ldots a_{n-1}) = c\int_0^\infty J_1(uc)\, J_0(ub)\, J_0(ua)\, J_0\, ua_1)\ldots J_0(ua_{n-1})\, du$$

for the probability, that the path ends inside the circle.

the coordinates p, q are surpassing the total length of the path. Then the probability becomes a certainty and it follows that

$$\frac{\pi^2}{4} = \int_0^\infty\int_0^\infty dv\, dw \frac{\sin pv}{v} \cdot \frac{\sin qw}{w} \cdot f(\sqrt{v^2 + w^2})$$

with the condition

$$p \text{ and } q > a + a_1 + \ldots + a_{n-1}.$$

In the general case of the rectangle the probability $W_n(R)$ is independent of q, as soon as its length is superior to that of the path.

Assuming this to be the case, we remark that the value of the slightly transformed integral

$$W_n(R) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty dv\, dw \frac{\sin pv}{v} \cdot \frac{\sin w}{w} \cdot f\left(\sqrt{v^2 + \frac{w^2}{q^2}}\right)$$

remains unaltered, when q increases indefinitely, and we conclude that

$$\lim_{q=\infty} W_n(R) = \frac{4}{\pi^2}\int_0^\infty \frac{\sin pv}{v} f(v) dv \,.\int_0^\infty \frac{\sin w}{w} dw = \frac{2}{\pi}\int_0^\infty \frac{\sin pv}{v} f(v) dv.$$

Thus we have solved another modification of PEARSON's problem, for half the result, added to $\frac{1}{2}$, expresses the probability

$$W_n(F) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin pv}{v} f(v)\, dv\,,$$

that the rambler, starting on his walk at a distance p of a straight frontier F, after walking along n stretches, will arrive at that side of the frontier he came from [1]).

As before we are enabled in a particular case by the problem itself to assign the value of the integral. If we suppose that the rambler cannot reach the frontier, that is, if we take

$$p > a + a_1 + \ldots + a_{n-1}\,,$$

the probability becomes a certainty and we find

1) Obviously the probability $W_n(F)$ might have been derived from the probability $W_{n+1}(\omega + p : \omega\, a\, a_1 \ldots a_{n-1})$ by making ω indefinitely large. Therefore we may conclude that

$$\lim_{\omega=\infty} (\omega+p)\int_0^\infty J_1(u\overline{\omega+p})\, J_0(u\omega) f(u) du = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin vp}{v} f(v) dv.$$

$$\frac{\pi}{2}=\int_0^\infty \frac{\sin vp}{v} J_0(va) J_0(va_1)\ldots J_0(va_{n-1})\, dv\,.$$

In the case $n=1$, this is a known result to which another may be added, if we take $a>p$. When the single stretch a is inclined to the frontier under an angle less than

$$arc\sin\frac{p}{a},$$

the rambler remains at the same side and, all directions of the stretch being equally possible, we have

$$W_1(F)=\frac{1}{\pi}\left(\frac{\pi}{2}+arc\sin\frac{p}{a}\right),$$

hence

$$arc\sin\frac{p}{a}=\int_0^\infty \frac{\sin vp}{v} J_0(va)\, dv\,.$$

Mathematics. — *"A definite integral of* KUMMER". By Prof. W. KAPTEYN.

In CRELLE's Journal, Vol. 17, KUMMER has determined the value of the integral

$$U_p=\int_0^\infty e^{-x-\frac{b^2}{x}} x^p\, dx\,,$$

supposing b^2 to represent a positive quantity and p not an integer. He finds:

$$U_p=\Gamma(p+1) f(-p, b^2)+\Gamma(-p-1)\, b^{2p+2} f(p+2.\, b^2),$$

where

$$f(p,x)=1+\frac{x}{1!p}+\frac{x^2}{2!p(p+1)}+\frac{x^3}{3!p(p+1)(p+2)}+\ldots$$
$$+\ldots=\sum_{s=0}^{\infty}\frac{x^s}{s!p(p+1)..(p+s-1)}\,.$$

In the following pages we propose to study this integral for the case that p represents a positive integer, and at the same time to show that there is a simple connection between this integral and the integral

$$V_p=\int_b^\infty e^{-x-\frac{b^2}{x}} x^p\, dx,$$

where b is supposed to be positive.