Mathematics. — "*An article on the knowledge of the tetrahedral complex.*" By Dr. Z. P. BOUMAN. (Communicated by Prof. JAN DE VRIES).

§ 1. When for an arbitrary ray out of a tetrahedral complex P_i represents the point of intersection with the face $A_k A_l A_m$ of the tetrahedron, then

$$R p_2 p_5 + p_3 p_6 = 0,$$

where R represents the given anharmonic ratio of the complex and p_i $(i = 1 \, . \, . \, 6)$ are the PLUCKER coordinates of lines.

By using the condition necessary for each ray of the complex, namely

$$p_1 p_4 + p_2 p_5 + p_3 p_6 = 0$$

the equation of the complex becomes

$$A p_1 p_4 + B p_2 p_5 + C p_3 p_6 = 0,$$

where the anharmonic ratio is given by

$$R = \frac{B - A}{C - A}.$$

A given tetrahedral complex can always transform itself projectively into another one with the same anharmonic ratio in regard to the faces of the rectangular system of coordinates and the plane at infinity.

§ 2. After having executed this transformation we can examine whether a surface with two independent parameters can be found in such a manner that the normals to be erected in an arbitrary point on the ∞^1 number of surfaces passing through that point, are rays of the given tetrahedral complex.

To this end we make the two determining points to lie infinitely close to each other on each ray of the complex, so that each ray is determined by one point (x, y, z) and the direction (dx, dy, dz) in that point. The coordinates of lines now take the form:

$$p_1 = x\,dy - y\,dx, \quad p_2 = y\,dz - z\,dy, \quad p_3 = z\,dx - x\,dz,$$
$$p_4 = -dz, \qquad p_5 = -dx, \qquad p_6 = -dy.$$

So the equation for the complex becomes:

$$A\,(x\,dy - y\,dx)\,dz + B\,(y\,dz - z\,dy)\,dx + C\,(z\,dx - x\,dz)\,dy = 0.$$

If now every ray of the complex is to be at right angles to a surface $z = f(x, y)$, then we have for each ray in each point of the surface:

$$dx : dy : dz = p : q : -1 ,$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

So the differential equation of the surface becomes:

$$-pqz(B - C) + yp(A - B) + xq(C - A) = 0$$

or

$$z - \frac{x}{p}\frac{1}{R-1} - \frac{y}{q}\frac{R}{1-R} = 0 .$$

The complete integral with two parameters C and C_1 becomes:

$$z = \pm \sqrt{\frac{1}{R-1}}\sqrt{x^2 - C} \pm \sqrt{\frac{R}{1-R}}\sqrt{y^2 - C_1} .$$

It represents a surface of order four.

It is evident out of the equation that for $R = \frac{1}{R'}$ the surface remains the same; only the X- and the Y-axes have been interchanged. (This is geometrically immediately made clear). So we have but to examine the surface for, let us say, $R > 1$.

§ 3. It must be possible to find the equation of the cone of the complex in a definite point out of the equation of the surface because that cone is the locus of the normals to the ∞^1 number of surfaces, passing through the point under consideration. If α, β, γ represent the cosines of direction of a ray of the complex in the point x_1, y_1, z_1 then $p = -\frac{\alpha}{\gamma}, q = -\frac{\beta}{\gamma}$.

Substituting this in the differential equation and eliminating α and β by means of the equations of the ray of the complex, namely

$$\frac{x - x_1}{\alpha} = \frac{y - y_1}{\beta} = \frac{z - z_1}{\gamma},$$

we find for the cone of the complex:

$$(R-1)\, z_1 (x - x_1)(y - y_1) - R y_1 (x - x_1)(z - z_1) + x_1 (y - y_1)(z - z_1) = 0.$$

The planes of the coordinates forming the singular surface of the complex, the cone of the complex must degenerate for each point of one of these planes. For the point $P(x_1, y_1 = 0, z_1)$ the cone breaks up into $y = 0$ and into $x_1 z + (R-1) z_1 x = R z_1 x_1$, i. e. a plane passing through P and parallel to the Y-axis. This plane is at right angles to OP, if this line has for equation $z = \pm x \sqrt{\frac{1}{R-1}}$. (Comp. § 4).

§ 4. The drawing of the surfaces to be found offers no difficulties.

For $R > 1$ (§ 2) we must take C_1 positive and then we have to distinguish the cases $C \gtreqless 0$.

So for $C > 0$ the surface consists of two separated parts connected by points forming parts of a double conic in the XOY-plane. The planes $x = \pm \sqrt{C}$ touch both parts according to equal ellipses and no points lie between with $z > 0$.

The section with the XOZ-plane consists of two hyperbolae with centres $\left(z = \pm \sqrt{\frac{RC_1}{R-1}}\right)$ on the Z-axis. At infinity they are connected twice, and intersect each other in the points of intersection of the double conic with the X-axis. The hyperbolae coincide in the planes $y = \pm \sqrt{C_1}$, where the common vertex of the double conic is lying.

C becoming smaller, the two parts of the surface approach each other and for $C = 0$ the conics meet in the planes $x = \pm \sqrt{C}$ The surface becomes a ruled surface, so it breaks up into two cylinders with axes in the XOZ-plane.

The axes have for equation $z = \pm x \sqrt{\frac{1}{R-1}}$. (Comp. § 3). The section perpendicular to these axes is a circle which is in accordance with the signification of the axes as found in § 3.

§ 5. It is known that the normals of a system of similar, concentric ellipsoids form a tetrahedral complex [1]). So this system must be a particular integral of the above-mentioned differential equation.

Let us put $C = gC_1 + h$ (g and h being constants) and let us operate in the ordinary way; we find C and C_1 as functions of the variables out of:

$$y^2 - C_1 = R\frac{gy^2 + h - x^2}{g(g+R)},$$

$$x^2 - C = -g\frac{gy^2 + h - x^2}{g+R}.$$

Substitution in the complete integral furnishes:

$$z^2\frac{g(1-R)}{g+R} - gy^2 + x^2 = h.$$

Let us put in this equation $g = -\frac{a^2}{b^2}$, and let c be the axis along the Z-axis; we shall then find if we take a^2 positively

1) Dr. J. DE VRIES: On a special tetrahedal complex. Proceedings of Febr. 25 1905, Vol. XIII, pages 572—577.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=h', \text{ with } R=\frac{a^2-c^2}{b^2-c^2}.$$

Likewise $\left(g=-\frac{a^2}{b^2},\ a^2 \text{ negative}\right)$ the system of hyperboloids with two sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=h', \text{ with } R=\frac{a^2+c^2}{b^2+c^2}$$

and also $\left(g=\frac{a^2}{b^2},\ a^2 \text{ positive}\right)$ the system of hyperboloids with one sheet

$$\frac{z^2}{c^2}+\frac{x^2}{a^2}-\frac{y^2}{b^2}=h', \text{ with } R=\frac{c^2-a^2}{c^2+b^2}.$$

§ 6. The "curves of a complex" are curves whose tangents are rays of the complex. The coefficients of direction (α, β, γ) in a definite point (x, y, z) must therefore be proportional to

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ -1,$$

of one of those surfaces through that point. From this ensues that $p=-\frac{\alpha}{\gamma}$ and $q=-\frac{\beta}{\gamma}$, whilst x, y, z, p and q must satisfy the equation:

$$z-\frac{x}{p}\frac{1}{R-1}-\frac{y}{q}\frac{R}{1-R}=0.$$

So the quantities $x, y, z, \alpha, \beta, \gamma$ must satisfy:

$$\frac{z}{\gamma}-\frac{y}{\beta}\frac{R}{R-1}+\frac{x}{\alpha}\frac{1}{R-1}=0.$$

Let a curve of the complex be given by:

$$x=f_1(s),\quad y=f_2(s),\quad z=f_3(s),$$

where s need not of necessity represent the length of the arc, then:

$$\frac{f_3(s)}{f_3'(s)}+\frac{f_2(s)}{f_2'(s)}\frac{R}{1-R}+\frac{f_1(s)}{f_1'(s)}\frac{1}{R-1}=0.$$

Amongst others all curves for all values of p to be represented by

$$x=\lambda(l+s)^p,\quad y=\mu(m+s)^p,\quad z=\nu(n+s)^p$$

satisfy this equation if only

$$\frac{l-n}{m-n}=R,$$

which condition can be satisfied by putting $l=B$, $m=C$, $n=A$.

For $p = -1$ these are twisted cubics. If we bring these through a point (x_1, y_1, z_1) the ∞^1 curves all lie on the cone of the complex of this point. This holding for each point, the bisecants (and not only the tangents) are rays of the complex.

Indeed, all the twisted cubics pass through the vertices of our tetrahedron and the four planes passing through a bisecant and these four points have thus a constant anharmonic ratio. From this ensues that the bisecants intersect the four planes of coordinates in the same anharmonic ratio.

For $p = 1$ we have the rays of the complex themselves.

For $p = 2$ we have conics which can be nothing but conics of the complex, e.g. for $s = -l$ the curve touches the plane YOZ, etc.

For $p = 3$ we have twisted cubics whose bisecants are not rays of the complex, etc.

In general the tangents to the "curves of a complex" lie always in linear congruences belonging to the tetrahedral complex. For such a tangent namely we have

$$(l+s)\frac{dx}{x} = (m+s)\frac{dy}{y} = (n+s)\frac{dz}{z}.$$

From this ensues among others:

$$(n+s)\frac{dz}{z} = \frac{(l+s)\,dx + k\,(m+s)\,dx}{x+ky}. \quad (k \text{ an arbitrary constant.})$$

This is evidently always satisfied by rays of the complex, satisfying at the same time:

$$xdz - zdx = k\,(zdy - ydz) \quad \text{and} \quad kdy = -Rdx,$$

for which we can write in coordinates of lines:

$$p_3 = kp_2 \quad \text{en} \quad -kp_6 = Rp_5.$$

These satisfy the equations of the tetrahedral complex and lie in congruences; the two linear complexes determining such a congruence, are themselves special, and the position of their axes is evident from their equation.

§ 7. Finally it proves to be simple to bring in equation the curves which are drawn on an arbitrary surface in such a way that the cone of the complex touches the surface in each point of the curve.

Let the surface be $f(x, y, z) = 0$ and the ray of the complex $\frac{x-x_1}{\alpha} = \frac{y-y_1}{\beta} = \frac{z-z_1}{\gamma}$, passing through the point x_1, y_1, z_1 of the surface.

A ray of the complex in the tangential plane must satisfy

$$\alpha \frac{\partial f}{\partial x_1} + \beta \frac{\partial f}{\partial y_1} + \gamma \frac{\partial f}{\partial z_1} = 0,$$

and further according to the differential equation

$$(R-1)\, z_1\, \alpha\beta - R\, y_1\, \alpha\gamma + x_1\, \beta\gamma \text{ must be equal to } 0.$$

The two rays of the complex in the tangential plane have but to be made to coincide. The condition is:

$$-4\,R\,(R-1)\, z_1\, y_1\, f_2 f_3 = [-(R-1)\, z_1 f_3 + R\, y_1 f_2 + f_1\, x_1]^2,$$

where f_1, f_2, f_3 represent the differential quotients of f according to x, y and z respectively, whilst analogous relations are easy to deduce.

From this ensues that the required curve is the intersection of

$$f(x, y, z) = 0$$

and

$$-4\,R\,(R-1)\, z\, y\, f_2 f_3 = [-(R-1)\, z f_3 + R\, y f_2 + f_1\, x]^2.$$

Without entering into further details I only wish to observe that when $f(x, y, z) = 0$ represents a plane, the curve can be nothing but the conic of the complex. From the above mentioned equations we therefore find a parabola (the conic of the complex touches the tetrahedron plane at infinity) touching the three planes of coordinates of the rectangular system of axes.

Physiology. — *"On the excretion of creatinin in man"*. By C. A. PEKELHARING. Report of a research made by C. J. C. VAN HOOGENHUYZE and H. VERPLOEGH.

As the muscle tissue in herbivora as well as in carnivora always contains a not unimportant amount of creatin, and creatinin is daily excreted with the urine it may be concluded, that creatin is formed as a product of metabolism in the muscles, and having entered the blood is at least for a part excreted by the kidneys in the form of the anhydride, creatinin.

But no agreement has been obtained about the question whether the forming of creatin is bound to the labour, the contracting of the muscles. To answer that question, researches have been made whether the amount of creatinin excreted by the kidneys augments after muscular labour. Different investigators have obtained different results. VAN HOOGENHUYZE and VERPLOEGH have resumed the research anew, using a new method to determinate the amount of creatinin in the urine, which was published some time ago by FOLIN [1]). The

[1]) Zeitschr. f. Physiol. Chemie, Bd. XLI, S. 223.