KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN TE AMSTERDAM.

PROCEEDINGS OF THE MEETING

of Saturday October 28, 1905.

(Translated from: Verslag van de gewone vergadering der Wis- en Natuurkundige Afdeeling van Zaterdag 28 October 1905, Dl. XIV).

CONTENTS.

The following papers were read:

Mathematics. — "*Central Projection in the space of* LOBATSCHEFSKY". (1st part). By Prof. H. DE VRIES. (Communicated by Prof. J. CARDINAAL).

(Communicated in the meeting of September 30, 1905).

1. Let an arbitrary plane τ be given in hyperbolic space; let the perpendicular be erected in an arbitrary point O_1 of τ, and let finally an arbitrary point O be taken on this perpendicular. We can now ask what we can notice if we project the figures of space out of O as a centre of projection on τ as a plane of projection or picture plane, and inversely, how the exact position and situation of the figures in space can be determined by means of their projections.

In the following a few observations will be given on these two questions.

Let us suppose an arbitrary plane ε through the line OO_1, standing therefore at right angles to τ; in that plane we can draw through O two straight lines p_1, p_2 parallel to the line of intersection e of ε and τ passing through O_1, therefore also parallel to τ itself. The angles formed by p_1 and p_2 with OO_1 are equal; they are both acute, and their amount is a function of the distance $OO_1 = d$. LOBATSCHEFSKY has called each of these two angles the *parallel angle* [1]) belonging to the distance d, and has indicated them by $\Pi_{(d)}$; if d is given, the parallel angle is found out of the relation

$$tg\, {}^1/_2\, \Pi_{(d)} = e^{-d},$$

in which for the number e the basis of the natural logarithmic system may be taken, if only the unity of length by which OO_1 is measured be taken accordingly [2]). As far as the range of values of $\Pi_{(d)}$ is concerned, I only observe that the parallel angle $= {}^1/_2\,\pi$ for $d = 0$, decreasing and tending to 0 if d increases and tends to ∞.

If the plane ε rotates round OO_1, then p_1 and p_2 will describe a cone of revolution round OO_1 as axis; this cone is the locus of all straight lines through O parallel to τ, and distinguishes itself in many respects, in form and properties, from the cone of revolution of Euclidian Geometry; the plane τ is an asymptotic plane, to which its surface tends unlimited, and from the symmetry with respect to O follows easily that another plane τ^* like this exists, also placed perpendicularly on OO_1, but in the point $O_1{}^*$ situated symmetrically to O_1 with respect to O. So the cone is entirely included between the two planes τ and τ^*, and these two planes having not a single point in common (neither at finite nor at infinite distance), are *divergent;* however, they possess the common perpendicular $O_1O_1{}^*$, and their shortest distance is $2d$. The cone discussed here will be called for convenience, sake the parallel cone $\varkappa$ belonging to the point O.

2. The parallel cone divides the space into three separate parts; let us call those two parts, inside which is the axis OO_1, the interior of the cone, the remaining part the exterior; it is then easy to see that the points of space behave differently with respect to their projectability according to their lying inside or outside the cone; for a point P inside the cone the projecting ray OP forms with

1) F. ENGEL: „N. I. LOBATSCHEFSKY. Zwei geometrische Abhandlungen". Leipzig, TEUBNER, 1899, p. 167.

2) F. ENGEL, l. c. p. 214.

OO_1 [1]) an acute angle smaller than the parallel angle, from which ensues that OP (or perhaps the prolongation of PO over O) must meet the picture plane; the point of intersection P' is the central projection of P. However, for P outside the cone the acute angle between OP and OO_1 is greater than the parallel angle; so now OP is divergent with respect to τ, from which ensues that points outside the cone possess no projections at all; points on the cone on the contrary do, but these projections lie at infinity.

From the fact that points outside the parallel cone are not projectible we need not infer that these points, cannot be determined by Central Projection; if such a point is regarded let us say as the point of intersection of two straight lines, and if these *can* be projected by Central Projection, their point of intersection will also be determined in this indirect way.

3. Let a straight line l be at right angles to τ in a point D of τ. As the line OO_1 is also perpendicular to τ, it is possible to bring a plane through l and OO_1, the trace of which connects O_1 with D. This plane will intersect the cone $\varkappa$ in two generatrices, p_1, p_2; we now assume that l cuts these two lines in two points P_1 and P_2, and — to fix our thoughts — that P_1 lies between D and P_2. The line l possesses two points at infinity, $V_{1\infty}$, $V_{2\infty}$, which both lie inside the parallel cone; let us suppose that $V_{1\infty}$ lies under the picture plane and $V_{2\infty}$ above it, then the succession of the points on l is this: $V_{1\infty}$, D, P_1, P_2, $V_{2\infty}$.

The projecting ray $OV_{1\infty}$ cuts τ in a point V'_1 of e lying between D and O_1; we shall call it the first *vanishing point* of l. In like manner the ray $OV_{2\infty}$ prolonged over O will cut the line e in a point V'_2 lying in such a way that O_1 lies between V'_1 and V'_2; we shall call V'_2 the second vanishing point of l. The point O_1 does not lie in the middle between V'_1 and V'_2, on the contrary it is closer to V'_1; if namely we let down the perpendicular OS out of O to l a quadrangle is formed with three right angles, namely at O_1, D and S, and from this ensues that the fourth $\angle SOO_1$ is acute. Now OS is the bisectrix of $\angle V_{1\infty}OV_{2\infty}$, and therefore the perpendicular in O on OS the bisectrix of $\angle V'_1OV'_2$; this perpendicular must be placed, as $\angle SOO_1$ is acute, between OO_1, and OV'_2, and from this ensues $\angle V'_1OO_1 < \angle V'_2OO_1$. If we now let the rectangular triangle V'_1OO_1 rotate about the side OO_1 till it lies on the triangle V'_2OO_1, then we can immediately find that $O_1V'_1 < O_1V'_2$.

[1]) By OO_1 we understand the straight line prolonged at both ends unlimitedly, by OP however a semi-ray starting from O.

It is clear that the central projection l' of l coincides with the trace e of the projecting plane $Ol = \varepsilon$, and at the same time that l is determined by its point of intersection and by *one* of the two vanishing points; the second will be found by letting down the perpendicular OS out of O to l, and by setting off at the other side of OS an angle equal to the parallel angle, formed by OS and the only existing ray parallel to l. But further can be remarked that l is also determined by its two vanishing points, or what comes to the same by its two points at infinity; to find l we should but have to bisect the angle formed by the two projecting parallel rays of l, to mark on the bisecting line a segment OS corresponding to $^1/_2 \angle V_{1\infty} O V_{2\infty}$ as parallel angle, and to erect the perpendicular in S on OS.

The line l is divided into four segments by its two points at infinity, its point of intersection and its two points P_1, P_2 (whose projections lie at infinity), and l' in like manner by its points at infinity $P'_{1\infty}$, $P'_{2\infty}$, the point D, and the two vanishing points V'_1, V'_2 of l; the connection between these different segments of l and l' is as follows. To the infinite segment $V_{1\infty} D$ corresponds the finite segment $V'_1 D$, and to the finite segment DP_1 the infinite segment $DP'_{1\infty}$; to the points between P_1 and P_2 no projections correspond, because the projecting rays of these points are divergent with respect to τ; to the infinite segment $P_2 V_{2\infty}$ on the contrary a segment of l' again corresponds, namely the infinite segment $P'_{2\infty} V'_2$. There now remain on l' only the points between the two vanishing points, to which also belongs O_1; to these no points of l correspond, their projecting rays being divergent with respect to l.

§ 4. If a line $l \perp \tau$ is to cut the surface of the parallel cone in two points, the length of DO_1 may not exceed a certain upper limit, so that the results just found do not hold for *all* lines $\perp \tau$. Let us again suppose through OO_1 an arbitrary plane ε, and let us now first regard OO_1 itself. If we let down out of O_1 on to p_2 the perpendicular $O_1 T$, then because p_2 is parallel to e, the angle $TO_1 P'_{2\infty}$ is the parallel angle belonging to $O_1 T$, and therefore angle $TO_1 O$ is smaller than this parallel angle, because $O_1 O$ cuts the line p_2 (namely in $\angle O$); and $OO_1 P'_2$ being equal to 90°, the parallel angle $TO_1 P'_{2\infty} > 45°$, and angle $TO_1 O < 45°$. If in ε we move l, first coinciding with OO_1, in such a way that it remains in D perpendicular to e, namely towards the side of $P'_{1\infty}$ (therefore from $P'_{2\infty}$), then the perpendicular DT on p_2 becomes continually greater, and so (see N°. 1) the parallel angle $TDP'_{2\infty}$ continually smaller; as soon as the perpendicular DT has attained such a length that the parallel angle corresponding to it is precisely 45°, the complement becomes 45° too, and therefore l parallel to p_2, but on the other side of DT compared to e; l will still intersect p_1 in a finite point P_1, for as it enters the triangle $OO_1P'_{1\infty}$ at D, does of course not contain the point $P'_{1\infty}$, and is divergent with reference to OO_1, it can leave the triangle only in a finite point of p_1; but it will cut p_2 in an point at infinity $P_{2\infty}$, being at the same time $V_{2\infty}$. So its projection consists of the segment of the line e of V'_1 over D to $P'_{1\infty}$, and the isolated point $P'_{2\infty}$ is equal to $V'_{2\infty}$; now too it is determined by two of the three points D, V'_1, $V'_{2\infty}$.

The point D lies at a certain distance r from O_1; if we describe a circle in τ about O_1 as centre and with r as radius, and if we erect in all points of that circle the perpendiculars on τ, a surface appears which may be called a cylinder of revolution, of which the circle just mentioned is the gorge line; the lines l ($\perp\tau$) lying inside that cylinder have two different vanishing points (with the exception of OO_1, whose projection is a single point), the lines l on the cylinder have a finite and an infinite vanishing point, and the lines l outside the cylinder miss the second vanishing point.

As for the shape of the cylinder it is easy to see, that the plane τ^* (see N°. 1) is an asymptotic plane; and τ itself being evidently a plane of orthogonal symmetry, the plane τ^{**} normal to OO_1 in the point O_1^{**} symmetrical to O_1^* with respect to τ will be a second asymptotic plane; so the distance of these two planes is $4d$.

5. In Euclidean Geometry the lines $\perp\tau$ are at the same time those which are parallel to OO_1, but in Hyperbolic Geometry this is different; here we have to regard the lines having in common with OO_1 the point $V_{1\infty}$ lying under the picture plane at infinity, and those having with OO_1 in common the point $V_{2\infty}$ lying above τ. A line l of the former kind lying in the vicinity of OO_1 has a picture point D, two points P_1, P_2, and a second point at infinity lying inside the cone $\varkappa$; its first vanishing point coincides with O_1, whilst the second lies on DO_1 in such a way, that O_1 lies between D and that point.

If the perpendicular OS let down out of O to l becomes continually larger, the first particularity appearing here is that l becomes parallel to the generatrix p_2 of cone $\varkappa$ lying in the plane Ol; then it is at right angles to the bisectrix of the obtuse angle formed by p_2 and OO_1. All lines having this property form an asymptotic cone of revolution [1]) with vertex $V_{1\infty}$, whilst τ^* is an asymptotic plane;

[1]) H. Liebmann, "Nichteuklidische Geometrie", Collection Schubert XLIX, page 63.

as base circle we can obtain a circle with finite radius in τ. For the generatrices of this cone the second point P_2 coincides with the second point *at infinity; so the projection consists of the infinite segment* $O_1 DP'_{1\infty}$ and the isolated second point at infinity of this line.

For lines l outside this cone this isolated point vanishes, and on account of this the second vanishing point; for its determination remain however D, and the first vanishing point O_1. Now however, the perpendicular OS still increasing, l can become parallel to p_1, and hence parallel to e or to τ; it is then at right angles to the bisectrix of the acute angle between p_1 and OO_1, as well as to that of the right angle between e and $O_1 V_{1\infty}$, which bisectrices are respectively divergent. All lines showing this property form a second asymptotic cone of revolution; for which however τ is now the asymptotic plane; they have a picture point at infinity, but are no less determined by this point and the first vanishing point O_1.

If l also lies outside this second cone, it becomes divergent with respect to τ, so it loses its picture point D; but now its second point at infinity lies again inside the cone $\varkappa$, which makes it projectible, so that in this case l has two vanishing points but no picture point; however, the two vanishing points are sufficient for its determination (see N°. 3). The originals at infinity corresponding to it are both under the picture plane; in connection with the preceding it would be preferable, in order to avoid confusion, to say that l has in this case two "first points at infinity" and therefore also two "first vanishing points".

The lines containing the second point at infinity $V_{2\infty}$ of OO_1 behave in like manner; we again find two asymptotic cones of revolution, one with the asymptotic plane τ, a second with the asymptotic plane τ^*, and we terminate with lines with two "second vanishing points" and without picture point.

Delft, September 1905.

Physiology. — *"A method for determining the osmotic pressure of very small quantities of liquid."* By Prof. H. J. HAMBURGER.

It not unfrequently happens that one wishes to know the osmotic pressure of normal or pathological somatic fluids of which no more than $^1/_2$ or $^1/_4$ cc. are available. I recently had such a case when an oculist asked me what should be the concentration of liquids used for the treatment of the eye. It seemed to me to be rational — and the investigations of MASSART [1]) justified this opinion — to prescribe

[1]) MASSART, Archives de Biologie **9** 1889, p. 335.