Physics. — *"On the radiation of heat in a system of bodies having a uniform temperature"*. By Prof. H. A. Lorentz.

(Communicated in the meetings of September and October 1905).

§ 1. A system of bodies surrounded by a perfectly black enclosure which is kept at a definite temperature, or by perfectly reflecting walls, will, in a longer or a shorter time, attain a state of equilibrium, in which each body loses as much heat by radiation as it gains by absorption, the intervening transparent media being the seat of an energy of radiation, whose amount per unit of volume is wholly determinate for every wave-length. The object of the following considerations is to examine somewhat more closely this state of things and to assign to each element of volume its part in the emission and the absorption. Of course, the most satisfactory way of doing this would be to develop a complete theory of the motions of electrons to which the phenomena may in all probability be ascribed. Unfortunately however, it seems very difficult to go as far as that. I have therefore thought it advisable to take another course, based on the conception of certain periodic electromotive forces acting in the elements of volume of ponderable bodies and producing the radiation that is emitted by these elements. If, without speaking of electrons, or even of molecules, we suppose such forces to exist in a matter continuously distributed in space, and if we suppose the emissivity of a black body to be known as a function of the temperature and the wave-length, we shall be able to calculate the intensity that must be assigned to the electromotive forces in question. The result will be a knowledge, not of the real mechanism of radiation, but of an imaginary one by which the same effects could be produced.

§ 2. For the sake of generality we shall consider a system of aeolotropic bodies. As to the notations used in our equations and the units in which the electromagnetic quantities are expressed, these will be the same that I have used in my articles in the Mathematical Encyclopedia. We may therefore start from the following general relations between the electric force $\mathfrak{E}$, the current $\mathfrak{C}$, the magnetic force $\mathfrak{H}$ and the magnetic induction $\mathfrak{B}$

$$rot\,\mathfrak{H} = \frac{1}{c}\,\mathfrak{C}, \quad \ldots \quad (1)$$

$$rot\,\mathfrak{E} = -\frac{1}{c}\,\dot{\mathfrak{B}}, \quad \ldots \quad (2)$$

In these formulae c denotes the velocity of light in the aether.

In the greater part of what follows, we shall confine ourselves to cases, in which the components of the above vectors and of others we shall have occasion to consider, are harmonic functions of the time with the frequency n. Then, the mathematical calculations can be much simplified if, instead of the real values of these components, we introduce certain complex quantities, all of which contain the time in the factor e^{int} and whose real parts are the values of the components with which we are concerned. If $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$ are complex quantities of this kind, relating in one way or another to the three axes of coordinates and in which the quantity e^{int} may be multiplied by complex quantities, the combination $(\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z)$ may be called a complex vector $\mathfrak{A}$ and $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$ its components.

By the real part of such a vector we shall understand a vector whose components are the real parts of $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$. It will lead to no confusion, if the same symbol is used alternately to denote a complex vector and its real part. It will also be found convenient to speak of the rotation and the divergence of a complex vector, and of the scalar product $(\mathfrak{A}, \mathfrak{B})$ and the vector product $[\mathfrak{A} . \mathfrak{B}]$ of two complex vectors $\mathfrak{A}$ and $\mathfrak{B}$, all these quantities being defined in the same way as the corresponding ones in the case of real vectors. E. g., we shall mean by the scalar product $(\mathfrak{A} . \mathfrak{B})$ the expression $\mathfrak{A}_x \mathfrak{B}_x + \mathfrak{A}_y \mathfrak{B}_y + \mathfrak{A}_z \mathfrak{B}_z$.

It is easily seen that, if $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{C}$ and $\mathfrak{B}$ are complex vectors, satisfying the equations (1) and (2), their real parts will do so likewise. The denominations electric force, etc. will be applied to these complex vectors as well as to the real ones.

One advantage that is gained by the use of complex quantities lies in the fact that now, owing to the factor e^{int}, a differentiation with respect to the time amounts to the same thing as a multiplication by in; in virtue of this the relation between $\mathfrak{E}$ and $\mathfrak{C}$ and that between $\mathfrak{H}$ and $\mathfrak{B}$ may be expressed in a simple form. Indeed, we may safely assume that, whatever be the peculiar properties of a ponderable body, the components of $\mathfrak{C}$ are connected to those of $\mathfrak{E}$ by three linear equations with constant coefficients, containing the components and their differential coefficients with respect to the time. In the case of the complex vectors, these equations may be written as linear relations between the components themselves; in other terms, one complex vector becomes a linear vector function of the other. A relation of this kind between two vectors $\mathfrak{A}$ and $\mathfrak{B}$ can always be expressed by three equations of the form

$$\mathfrak{B}_x = v_{11} \mathfrak{A}_x + v_{12} \mathfrak{A}_y + v_{13} \mathfrak{A}_z,$$
$$\mathfrak{B}_y = v_{21} \mathfrak{A}_x + v_{22} \mathfrak{A}_y + v_{23} \mathfrak{A}_z,$$
$$\mathfrak{B}_z = v_{31} \mathfrak{A}_x + v_{32} \mathfrak{A}_y + v_{33} \mathfrak{A}_z,$$

which we shall condense into the formula

$$\mathfrak{B} = (v)\, \mathfrak{A}.$$

According to this notation we may put $\mathfrak{C} = (p')\, \mathfrak{E}$, or, as is more convenient for our purpose,

$$\mathfrak{E} = (p)\, \mathfrak{C}, \quad \ldots \ldots \ldots \quad (3)$$

the symbol (p) containing a certain number of coefficients p which are determined by the properties of the body considered. As a rule, these coefficients are complex quantities, whose values depend on the frequency n.

As to the relation between $\mathfrak{B}$ and $\mathfrak{H}$, we shall put

$$\mathfrak{B} = (\mu)\, \mathfrak{H},$$

or

$$\mathfrak{H} = (q)\, \mathfrak{B} \quad \ldots \ldots \ldots \quad (4)$$

We have further to introduce an electromotive force which will be represented by a vector $\mathfrak{E}_e$, or by the real part of a complex vector $\mathfrak{E}_e$. The meaning of this is simply that the current $\mathfrak{C}$ is supposed to depend on the vector $\mathfrak{E} + \mathfrak{E}_e$ in the same way in which it depends on $\mathfrak{E}$ alone in ordinary cases, so that

$$\mathfrak{E} + \mathfrak{E}_e = (p)\, \mathfrak{C} \quad \ldots \ldots \ldots \quad (5)$$

Similarly, we may assume a magnetomotive force $\mathfrak{H}_e$, replacing (4) by

$$\mathfrak{H} + \mathfrak{H}_e = (q)\, \mathfrak{B} \quad \ldots \ldots \ldots \quad (6)$$

This new vector $\mathfrak{H}_e$ however, does not correspond to any really existing quantity; it is only introduced for the purpose of simplifying the demonstration of a certain theorem we shall have to use.

As to the coefficients we have taken together in the symbols (p) and (q), we shall suppose them to be connected with each other in the way expressed by

$$p_{12} = p_{21},\ p_{23} = p_{32},\ p_{31} = p_{13}, \quad \ldots \ldots \quad (7)$$

and

$$q_{12} = q_{21},\ q_{23} = q_{32},\ q_{31} = q_{13} \quad \ldots \ldots \quad (8)$$

The only case excluded by this assumption is that of a body placed in a magnetic field.

For isotropic bodies we may write, instead of (5) and (6),

$$\mathfrak{E} + \mathfrak{E}_e = p\, \mathfrak{C}, \quad \ldots \ldots \ldots \quad (9)$$
$$\mathfrak{H} + \mathfrak{H}_e = q\, \mathfrak{B}, \quad \ldots \ldots \ldots \quad (10)$$

with only one complex coefficient p and one coefficient q.

§ 3. Before coming to the problem we have in view, it is necessary to treat some preliminary questions. In the first place, we shall examine the vibrations that are set up in an unlimited homogeneous and isotropic body subjected to given electromotive and magnetomotive forces, changing with the frequency n. This problem is best treated by using the complex vectors.

We may deduce from (1)

$$rot\ rot\ \mathfrak{H} = \frac{1}{c}\, rot\, \mathfrak{D},$$

or

$$grad\ div\ \mathfrak{H} - \Delta\, \mathfrak{H} = \frac{1}{c}\, rot\, \mathfrak{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and similarly from (2)

$$grad\ div\ \mathfrak{E} - \Delta\, \mathfrak{E} = -\frac{1}{c}\, rot\, \dot{\mathfrak{B}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Again, always using the equations (1), (2), (9) and (10), we find

$$div\ \mathfrak{B} = 0 \quad , \quad div\ \mathfrak{D} = 0,$$

$$div\ \mathfrak{H} = -\, div\ \mathfrak{H}_e, \quad div\ \mathfrak{E} = -\, div\ \mathfrak{E}_e,$$

$$rot\ \mathfrak{D} = \frac{1}{p}(rot\ \mathfrak{E} + rot\ \mathfrak{E}_e) = -\frac{1}{pc}\dot{\mathfrak{B}} + \frac{1}{p}\, rot\ \mathfrak{E}_e$$

$$= -\frac{1}{pqc}(\dot{\mathfrak{H}} + \dot{\mathfrak{H}}_e) + \frac{1}{p}\, rot\ \mathfrak{E}_e,$$

$$rot\ \dot{\mathfrak{B}} = \frac{1}{q}(rot\ \dot{\mathfrak{H}} + rot\ \dot{\mathfrak{H}}_e) = \frac{1}{qc}\dot{\mathfrak{D}} + \frac{1}{q}\, rot\ \dot{\mathfrak{H}}_e$$

$$= \frac{1}{pqc}(\dot{\mathfrak{E}} + \dot{\mathfrak{E}}_e) + \frac{1}{q}\, rot\ \dot{\mathfrak{H}}_e,$$

so that (11) and (12) become

$$\Delta\, \mathfrak{H} - \frac{1}{pqc^2}\dot{\mathfrak{H}} = -\, grad\ div\ \mathfrak{H}_e + \frac{1}{pqc^2}\dot{\mathfrak{H}}_e - \frac{1}{pc}\, rot\ \mathfrak{E}_e,$$

$$\Delta\, \mathfrak{E} - \frac{1}{pqc^2}\dot{\mathfrak{E}} = -\, grad\ div\ \mathfrak{E}_e + \frac{1}{pqc^2}\dot{\mathfrak{E}}_e + \frac{1}{qc}\, rot\ \dot{\mathfrak{H}}_e.$$

The solution of these equations may be put in a convenient form by means of two auxiliary vectors $\mathfrak{A}$ and $\mathfrak{Q}$. If these are determined by

$$\Delta\, \mathfrak{A} - \frac{1}{pqc^2}\dot{\mathfrak{A}} = -\, \mathfrak{E}_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\Delta\, \mathfrak{Q} - \frac{1}{pqc^2}\dot{\mathfrak{Q}} = -\, \mathfrak{H}_e, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we shall have

$$\mathfrak{H} = grad\ div\ \mathfrak{Q} - \frac{1}{pqc^2}\dot{\mathfrak{Q}} + \frac{1}{pc}\, rot\ \mathfrak{A}, \quad . \quad . \quad . \quad . \quad (15)$$

$$\mathfrak{E} = grad\ div\ \mathfrak{A} - \frac{1}{pqc^2}\dot{\mathfrak{A}} - \frac{1}{qc}\, rot\ \dot{\mathfrak{Q}} \quad . \quad . \quad . \quad . \quad (16)$$

Finally, putting

$$v^2 = i\, p\, q\, n\, c^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

we get instead of (13) and (14)

$$\Delta\, \mathfrak{A} - \frac{1}{v^2}\ddot{\mathfrak{A}} = -\, \mathfrak{E}_e,$$

$$\Delta\, \mathfrak{Q} - \frac{1}{v^2}\ddot{\mathfrak{Q}} = -\, \mathfrak{H}_e,$$

for the solution of which we may take

$$\mathfrak{A} = \frac{1}{4\pi}\int \frac{1}{r}\, \mathfrak{E}_e\left(t - \frac{r}{v}\right) d\,S, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\mathfrak{Q} = \frac{1}{4\pi}\int \frac{1}{r}\, \mathfrak{H}_e\left(t - \frac{r}{v}\right) d\,S. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Here $d\,S$ denotes an element of volume situated at a distance r from the point for which we wish to calculate $\mathfrak{A}$ and $\mathfrak{Q}$, and the index $\left(t - \frac{r}{v}\right)$ means that, in the expressions representing $\mathfrak{E}_e$ and $\mathfrak{H}_e$ for that element of volume, t is to be replaced by $t - \frac{r}{v}$.

The algebraic sign of v is left indeterminate by (17). We shall choose it in such a way that our formulae represent a propagation of vibrations *issuing* from the elements of volume in which $\mathfrak{E}_e$ and $\mathfrak{H}_e$ are applied.

For aether we have $q = 1$ and, as may easily be shown $\frac{1}{p} = i n$, $v = c$.

§ 4. We have next to establish the equation of energy. The calculations required for this purpose, as well as those we shall have to perform later on, may be much simplified, if we replace all discontinuities at the limit of two bodies by a gradual transition from one to the other; this may be done without loss of generality, because, in our final results, the thickness of the boundary layers may be made to become infinitely small. A further simplification is obtained by leaving out of consideration the imaginary magnetomotive forces, and by supposing the coefficients μ and q to be real. The coefficients p_{11}, p_{12} etc. however will always be considered as complex quantities. We shall decompose them into their real parts, which we shall denote by α_{11}, α_{12}, etc., and their imaginary parts, for which we shall write $-i\,\beta_{11}, -i\,\beta_{12}$, etc., so that $p_{11} = \alpha_{11} - i\,\beta_{11}$, etc.

28

The equation (5) now becomes

$$\mathfrak{E} + \mathfrak{E}_e = (a)\,\mathfrak{C} - i\,(\beta)\,\dot{\mathfrak{C}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

or, if we define a new vector $\mathfrak{D}$ by means of the equation

$$\mathfrak{C} = \dot{\mathfrak{D}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\mathfrak{E} + \mathfrak{E}_e = (a)\,\mathfrak{C} + n\,(\beta)\,\mathfrak{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In the deduction of the equation of energy we have to understand by $\mathfrak{E}$, $\mathfrak{E}_e$, $\mathfrak{H}$ and $\mathfrak{D}$ the real vectors. For these we have the formulae (1), (2) and (21), and besides, since q, a and β are real, the relations (4) and (22).

From (1) and (2) we may draw immediately

$$c\,\{(\mathfrak{H}\,.\,rot\,\mathfrak{E}) - (\mathfrak{E}\,.\,rot\,\mathfrak{H})\} = -(\mathfrak{H}\,.\,\dot{\mathfrak{B}}) - (\mathfrak{E}\,.\,\mathfrak{C}),$$

the left-hand member of which is

$$div\,\mathfrak{S},$$

if we define the vector $\mathfrak{S}$ by the equation

$$\mathfrak{S} = c\,[\mathfrak{E}\,.\,\mathfrak{H}], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

i.e., if we understand by it the vector product of $\mathfrak{E}$ and $\mathfrak{H}$, multiplied by c.

In the right-hand member we have in the first place

$$(\mathfrak{H}\,.\,\dot{\mathfrak{B}}) = \frac{1}{2}\frac{\partial}{\partial t}\,(\mathfrak{H}\,.\,\mathfrak{B}),$$

as may be seen from (4), if (8) is taken into account, and further, in virtue of (7), (21) and (22),

$$(\mathfrak{E}\,.\,\mathfrak{C}) = (\,(a)\,\mathfrak{C}\,.\,\mathfrak{C}) + \frac{1}{2}\,n\,\frac{\partial}{\partial t}\,(\,(\beta)\,\mathfrak{D}\,.\,\mathfrak{D}) - (\mathfrak{E}_e\,.\,\mathfrak{C}).$$

Our equation therefore takes the following form, in which the meaning of the different terms is at once apparent,

$$(\mathfrak{E}_e\,.\,\mathfrak{C}) = (\,(a)\,\mathfrak{C}\,.\,\mathfrak{C}) + \frac{1}{2}\,n\,\frac{\partial}{\partial t}(\,(\beta)\,\mathfrak{D}\,.\,\mathfrak{D}) + \frac{1}{2}\frac{\partial}{\partial t}\,(\mathfrak{H}\,.\,\mathfrak{B}) + div\,\mathfrak{S}.$$

The first member represents the work done by the electromotive force per unit of volume and unit of time; in the second member

$$w = (\,(a)\,\mathfrak{C}\,.\,\mathfrak{C}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

is the expression for the quantity of heat that is developed per unit of space and unit of time. Further, $\frac{1}{2}\,(\mathfrak{H}\,.\,\mathfrak{B})$ is the magnetic and $\frac{1}{2}\,n\,(\,(\beta)\,\mathfrak{D}\,.\,\mathfrak{D})$ the electric energy, both reckoned per unit of volume.

The vector $\mathfrak{S}$ denotes the flow of energy, so that the amount of energy an element of volume dS loses by this flow is given by $div\,\mathfrak{S}\,dS$.

§ 5. We may now pass to a theorem which I have formerly proved in a somewhat more cumbrous and less general way. In order to arrive at it, we have to use the complex vectors, supposing at the same time the existence of magnetomotive forces; we have therefore to apply the formulae (5) and (6).

We shall consider *two* different states with the same frequency n, both of which can exist in the system of bodies. The symbols $\mathfrak{E}$, $\mathfrak{H}$, etc. will be used for one state and the corresponding symbols, distinguished by accents, for the other. We shall proceed in a way much like the operations of the last paragraph, with this difference however, that we shall now combine quantities relating to one state with quantities belonging to the other.

We shall start from the relation

$$c\,\{(\mathfrak{H}'.\,rot\,\mathfrak{E}) - (\mathfrak{E}.\,rot\,\mathfrak{H}')\} = -(\mathfrak{H}'.\,\dot{\mathfrak{B}}) - (\mathfrak{E}.\,\mathfrak{C}').$$

Here the expression on the left is equal to

$$c\,div\,[\mathfrak{E}.\,\mathfrak{H}']$$

and on the other side we may put

$$(\mathfrak{H}'.\,\dot{\mathfrak{B}}) = in\,(\mathfrak{H}'.\,\mathfrak{B}) = in\,(\,(q)\,\mathfrak{B}'.\,\mathfrak{B}) - (\mathfrak{H}'_e.\,\dot{\mathfrak{B}}),$$
$$(\mathfrak{E}.\,\mathfrak{C}') = (\,(p)\,\mathfrak{C}.\,\mathfrak{C}') - (\mathfrak{E}_e.\,\mathfrak{C}'),$$

so that we find

$$c\,div\,[\mathfrak{E}.\,\mathfrak{H}'] = -\,in\,(\,(q)\,\mathfrak{B}'.\,\mathfrak{B}) - (\,(p)\mathfrak{C}\,.\,\mathfrak{C}') + (\mathfrak{H}'_e.\,\dot{\mathfrak{B}}) + (\mathfrak{E}_e.\,\mathfrak{C}').$$

The theorem in question is a consequence of this formula and the corresponding one that is got by interchanging the quantities belonging to the two states; we have only to subtract one equation from the other. Since, by (8) and (7)

$$(\,(q)\,\mathfrak{B}'.\,\mathfrak{B}) = (\,(q)\,\mathfrak{B}.\,\mathfrak{B}') \text{ and } (\,(p)\,\mathfrak{C}.\,\mathfrak{C}') = (\,(p)\,\mathfrak{C}'.\,\mathfrak{C}),$$

we find in this way

$$c\,\{div\,[\mathfrak{E}.\,\mathfrak{H}'] - div\,[\mathfrak{E}'.\,\mathfrak{H}]\} = (\mathfrak{H}'_e\,\dot{\mathfrak{B}}) - (\mathfrak{H}_e.\,\dot{\mathfrak{B}}') + (\mathfrak{E}_e.\,\mathfrak{C}') - (\mathfrak{E}'_e.\,\mathfrak{C}).$$

We shall finally multiply this by an element of volume dS, and take the integral of both sides over the space within a closed surface σ. If we denote by n the normal to the latter, drawn outward, the result will be

$$c\int\{\,[\mathfrak{E}.\,\mathfrak{H}']_n - [\mathfrak{E}'.\,\mathfrak{H}]_n\,\}\,d\sigma = \int\{(\mathfrak{H}'_e.\dot{\mathfrak{B}}) - (\mathfrak{H}_e.\dot{\mathfrak{B}}') + (\mathfrak{E}_e.\mathfrak{C}') - (\mathfrak{E}'_e.\mathfrak{C})\}\,dS \quad (25)$$

§ 6. There are a number of cases in which the first member of this equation is zero.

a. E. g. we may suppose the system to be limited on all sides in such a way that it cannot exchange rays with surrounding bodies; we can realize this by enclosing the system in an envelop that is perfectly reflecting on the outside. If, under these circumstances, the surface σ surrounds that envelop, we may put in every point of it $\mathfrak{E} = 0$, $\mathfrak{E}' = 0$, $\mathfrak{H} = 0$, $\mathfrak{H}' = 0$.

b. If the envelop is made of a perfectly conducting material, both the electric force $\mathfrak{E}$ and the force $\mathfrak{E}'$ will be normally directed in every point of its inner surface. Consequently, if the latter is chosen for the surface σ, we shall have

$$[\mathfrak{E} \cdot \mathfrak{H}']_n = 0 \text{ and } [\mathfrak{E}' \cdot \mathfrak{H}]_n = 0.$$

c. Finally we may conceive a system lying in a finite part of space and surrounded by aether, into which it emits rays travelling outwards to infinite distance. Taking in this case for σ a sphere of infinite radius, we shall show that for each element $d\sigma$ the factor by which it is multiplied in the equation (25) vanishes. The direction of the axes of coordinates being indeterminate, it will suffice to prove this proposition for the point P in which the sphere is cut by a line drawn from the centre O in the direction of the axis of x.

Now, if we confine ourselves to those parts of $\mathfrak{E}$, $\mathfrak{H}$, $\mathfrak{E}'$ and $\mathfrak{H}'$ which are inversely proportional to the first power of OP, as may obviously be done, we may consider the state of things near the point P as a propagation of vibrations in the direction OP, the electric and magnetic force being perpendicular to that direction and to each other. Denoting by a and b, a' and b' certain complex quantities, we may write

$$\begin{aligned}
\mathfrak{E}_x &= 0,\ \mathfrak{E}_y = a e^{int}, & \mathfrak{E}_z &= b e^{int},\\
\mathfrak{H}_x &= 0,\ \mathfrak{H}_y = -\,b e^{int}, & \mathfrak{H}_z &= a e^{int},\\
\mathfrak{E}'_x &= 0,\ \mathfrak{E}'_y = a' e^{int}, & \mathfrak{E}'_z &= b' e^{int},\\
\mathfrak{H}'_x &= 0,\ \mathfrak{H}'_y = -\,b' e^{int}, & \mathfrak{H}'_z &= a' e^{int},
\end{aligned}$$

and we have at the point P, since in it the normal to the spherical surface is parallel to the axis of x,

$$[\mathfrak{E} \cdot \mathfrak{H}']_n - [\mathfrak{E}' \cdot \mathfrak{H}]_n = (\mathfrak{E}_y \mathfrak{H}'_z - \mathfrak{E}_z \mathfrak{H}'_y) - (\mathfrak{E}'_y \mathfrak{H}_z - \mathfrak{E}'_z \mathfrak{H}_y) = 0.$$

These considerations show that in many cases the equation (25) reduces to

$$\int \{(\mathfrak{E}'_e \cdot \mathfrak{C}) - (\mathfrak{H}'_e \cdot \dot{\mathfrak{B}})\}\, dS = \int \{(\mathfrak{E}_e \cdot \mathfrak{C}') - (\mathfrak{H}_e \cdot \dot{\mathfrak{B}}')\} dS \quad . \quad (26)$$

§ 7. It is particularly interesting to examine the effects produced by an electromotive or a magnetomotive force which is confined to an infinitely small space S. Let P be any point of this region, $\mathfrak{a}$ a real vector having everywhere the same direction h and the same magnitude $|\mathfrak{a}|$, and let us apply in all points of S an electromotive force $\mathfrak{a}\, e^{int}$. Then we shall say that there is an "electromotive action" at the point P in the direction h. We may represent it by the symbol

$$\mathfrak{a}\, \mathrm{S}\, e^{int}$$

and we may consider its intensity and its phase to be determined by the real part of $|\mathfrak{a}|\, \mathrm{S}\, e^{int}$.

In a similar sense we can also conceive a "magnetomotive action" existing in some point of the system.

These definitions being agreed upon, equation (26) leads to the following remarkable conclusions.

a. Let there be, in the first of the two cases we have distinguished in the preceding paragraph, an electromotive action $\mathfrak{a}\, \mathrm{S}\, e^{int}$ at the point P in the direction h, and in the second case an electromotive action $\mathfrak{a}'\, \mathrm{S}' e^{int}$ at the point P' in the direction h', there being in neither case a magnetomotive force. Then the integrals in (26) are to be extended to the infinitely small spaces S' and S and the result may be written in the form

$$(\mathfrak{a}' \cdot \mathfrak{C}_{P'})\, \mathrm{S}' = (\mathfrak{a} \cdot \mathfrak{C}'_P)\, \mathrm{S}\,,$$

if we represent by $\mathfrak{C}_{P'}$ the current produced in P' in the first case and by $\mathfrak{C}'_P$ the current existing in P in the second.

Hence, assuming the equality

$$|\mathfrak{a}|\, \mathrm{S} = |\mathfrak{a}'|\, \mathrm{S}',$$

we conclude that

$$\mathfrak{C}_{h'P'} = \mathfrak{C}'_{hP} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (27)$$

The full meaning of this appears, if we write the two quantities in the form

$$\mathfrak{C}_{h'P'} = \mu\, e^{i(nt+\nu)} \quad \text{and} \quad \mathfrak{C}'_{hP} = \mu'\, e^{i(nt+\nu')}.$$

Indeed, (27) requires that

$$\mu = \mu'\,,\ \nu = \nu'\,,$$

and we have the theorem:

If an electromotive action applied at a point P in the direction h produces in a point P' a current whose component in an arbitrarily chosen direction h' has the amplitude μ and the phase ν, an equal electromotive action taking place at the point P' in the direction h' will produce a current in P, whose component in the direction h has exactly the same amplitude μ and the same phase ν.

b. Without changing anything in the circumstances of the first case, we shall now assume, that in the second the vibrations are excited not by electromotive forces, but by a magnetomotive action $\mathfrak{a}'\, \mathrm{S}' e^{int}$, at the point P' in the direction h'. We then find

$$-(\mathfrak{a}' \cdot \dot{\mathfrak{B}}_{P'})\, \mathrm{S}' = (\mathfrak{a} \cdot \mathfrak{C}'_P)\, \mathrm{S},$$

and, if we put

$$|\alpha|\, S = |\alpha'|\, S',$$

$$-\mathfrak{H}_{h'P'} = \mathfrak{C}'_{hP}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

a theorem similar to the former.

§ 8. The absorption of rays being measured by the amount of heat developed, the expression (24), in which $\mathfrak{C}$ is the real current, will be often used in what follows. It may be replaced by

$$w = (\mathfrak{F} \cdot \mathfrak{C}),$$

if we write $\mathfrak{F}$ for the vector $(\alpha)\ \mathfrak{C}$, so that

$$\mathfrak{F}_x = \alpha_{11}\, \mathfrak{C}_x + \alpha_{12}\, \mathfrak{C}_y + \alpha_{13}\, \mathfrak{C}_z, \text{ etc.} \quad . \quad . \quad . \quad . \quad (29)$$

Now, by a well known theorem, the axes of coordinates may always be chosen in such a way that the coefficients $\alpha_{12}, \alpha_{23}, \alpha_{31}$ in these equations become zero. Denoting the remaining coefficients by $\alpha_1, \alpha_2, \alpha_3$, we have for the relation between $\mathfrak{F}$ and $\mathfrak{C}$

$$\mathfrak{F}_x = \alpha_1\, \mathfrak{C}_x, \quad \mathfrak{F}_y = \alpha_2\, \mathfrak{C}_y, \quad \mathfrak{F}_z = \alpha_3\, \mathfrak{C}_z,$$

and for the development of heat

$$w = \alpha_1\, \mathfrak{C}_x^2 + \alpha_2\, \mathfrak{C}_y^2 + \alpha_3\, \mathfrak{C}_z^2 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The directions we must give to the axes in order to obtain these simplifications, may properly be called the *principal directions;* in general, they will not be the same for different frequencies. This is due to the fact that the coefficients in (29) depend on the value of n.

It is also to be noticed that by this choice of the axes of coordinates, the coefficients $\beta_{12}, \beta_{23}, \beta_{31}$, and p_{12}, p_{23}, p_{31} will not, in general, be made to become zero.

In the case of an isotropic body we may take as principal directions any three directions perpendicular to each other.

§ 9. Thus far we have only prepared ourselves for our main problem. In the next paragraphs we shall first consider the absorption by a very thin plate surrounded by aether on both sides, and receiving in the normal direction a beam of rays. Combining the result with the ratio between the emissivity and the coefficient of absorption of a body, we shall be able to determine the amount of energy, radiated by the plate in a normal direction, and our next object will be to calculate the intensity we must ascribe to electromotive forces acting in the plate (§ 1), in order to account for that radiation. This will lead us to a general hypothesis concerning the electromotive forces acting in the elements of volume of a ponderable body and we shall conclude by showing that, if these electromotive forces were applied, the condition required for the equilibrium of radiation would always be fulfilled.

§ 10. Let the plate be homogeneous, with its faces parallel to the first and the second principal direction. We shall take these for the axes of x and y, placing the origin O in the front surface of the plate, i. e. in the surface exposed to the rays, and drawing the axis of z toward the outside. As has already been said, the absorption will be calculated by means of the formula (30); it will therefore be determined by the components of $\mathfrak{C}$ and by those of $\mathfrak{E}$, on which they depend. Now, our problem is greatly simplified, if we suppose the thickness Δ of the plate to be infinitely small and if, in calculating the absorption, we confine ourselves to quantities of the first order of magnitude with respect to Δ. The quantity w relating to unit volume, we may then neglect all infinitely small terms in $\mathfrak{C}$ and $\mathfrak{E}$; consequently, we need not attend to the changes of these vectors in the plate along a line perpendicular to its faces. Moreover, in virtue of the well known conditions of continuity, the values of $\mathfrak{E}_x$ and $\mathfrak{E}_y$ within the plate will be equal to those existing in the aether immediately before it; also, $\mathfrak{C}_z$ will be 0, because it is so in the aether. For $\mathfrak{E}_x$ and $\mathfrak{E}_y$ we may even take the values, existing in the incident beam, the reason for this being that the values belonging to the reflected rays, (the vibrations reflected at the two sides being taken together) are proportional to the thickness, if the plate is infinitely thin.

It is seen by these considerations that in the case of a given incident motion, $\mathfrak{C}_x$, $\mathfrak{C}_y$, $\mathfrak{E}_z$ are the only unknown quantities in the three equations connecting the components of $\mathfrak{E}$ and $\mathfrak{C}$. We need not, however, work out the solution of these equations.

Finally, it must be kept in mind that, in the case of harmonic vibrations, the mean value of w for a lapse of time comprising many periods is given by

$$w = \frac{1}{2} \{ \alpha_1\, (\mathfrak{C}_x)^2 + \alpha_2\, (\mathfrak{C}_y)^2 + \alpha_3\, (\mathfrak{C}_z)^2 \} \,, \quad . \quad . \quad . \quad (31)$$

if $(\mathfrak{C}_x), (\mathfrak{C}_y), (\mathfrak{C}_z)$ are the *amplitudes* of the components of the current.

§ 11. We shall in the first place assume that in the incident rays the electric force is parallel to the axis of x. Let its amplitude be a. Then, an element ω of the front surface will receive an amount of energy

$$\frac{1}{2} c a^2 \omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

per unit of time.

Within the plate, there will be electric currents in the directions of x and y. These will have amplitudes proportional to a, and for which we may therefore write:

$$(\mathfrak{E}_x) = f\,a \quad , \quad (\mathfrak{E}_y) = g\,a$$

denoting by f and g two factors, which it will be unnecessary to calculate. From (31) we deduce for the heat developed in the part $\omega\,\Delta$ of the plate,

$$\frac{1}{2}(\alpha_1 f^2 + \alpha_2 g^2)\, a^2\, \omega\, \Delta$$

and, dividing this by (32), for the coefficient of absorption

$$A = \frac{1}{c}(\alpha_1 f^2 + \alpha_2 g^2)\,\Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Our next step must be to obtain a formula for the emission. For this purpose we fix our attention on a surface-element ω' parallel to the plate and situated at a large distance r from it, at a point of OZ. The electric vibrations issuing from the plate may be decomposed in the first place into vibrations of different frequencies and in the second place into components parallel to OX and OY.

After having effected this decomposition, we may attend to the amount of energy travelling across ω' per unit of time, in so far as it belongs to vibrations having the first of the two directions and to frequencies lying between the limits n and $n + dn$. Now, if the plate were removed, and if instead of it a perfectly black body of the same temperature were placed behind an opaque screen with an opening coinciding with the element ω, the radiation might be represented by

$$\frac{k\,\omega\,\omega'\,dn}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

an expression which may also be regarded as indicating the ratio between the emissivity of a body of any kind under the said circumstances and its coefficient of absorption. The experimental investigations of these last years have led to a knowledge of the coefficient k for a wide range of temperatures and frequencies.

By KIRCHHOFF's law, the flow of energy across the element ω', originated by the part

$$\omega\,\Delta = S$$

of the plate, in so far as it is due to vibrations of the said direction and frequency, is found by multiplying (34) by (33). Its amount is therefore

$$\frac{k\,S\,(\alpha_1 f^2 + \alpha_2 g^2)\,\omega'\,dn}{c\,r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and we have now to account for this radiation by means of suitable electromotive forces applied to the plate.

§ 12. We shall first put the question what must be the amplitude a_1 of an electromotive force acting in the direction of OX with the frequency n, if this force is to produce, on account of the electric vibrations parallel to OX, a flow of energy

$$\frac{k\,S\,\alpha_1 f^2\,\omega'\,dn}{c\,r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

across the element ω' at the point P. Since this flow may be represented by

$$\frac{1}{2}\,c\,b^2\,\omega',$$

if b is the amplitude of $\mathfrak{E}_x$ at the point P, we must have

$$b = \frac{f}{cr}\sqrt{2k\,S\,\alpha_1\,dn}.$$

The amplitude of the current $\mathfrak{C}_x = \dot{\mathfrak{E}}_x$ must therefore be

$$\frac{nf}{cr}\sqrt{2k\,S\,\alpha_1\,dn} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

At this stage of our reasoning we may avail ourselves of the theorem of § 7, a. Indeed, if the electromotive force $\mathfrak{E}_{ex}$ in the part S of the plate must have the amplitude a_1 in order to call forth at the point P a current $\mathfrak{C}_x$ whose amplitude has the value (37), a_1 will also be the amplitude we must give to an electromotive force $\mathfrak{E}_{ex}$, acting in an element of volume S of the aether near P, if we wish to bring about by its action a current with the amplitude (37) in the plate. This is the condition by which we shall determine the value of a_1.

§ 13. The solution is readily obtained by means of the formulae (18) and (16). If, in an element of volume S of the aether, $\mathfrak{E}_{ex} = a_1\, e^{int}$, $\mathfrak{E}_{ey} = 0$, $\mathfrak{E}_{ez} = 0$, we shall have

$$\mathfrak{A}_x = \frac{a_1\,S}{4\,\pi\,r}\,e^{in\left(t - \frac{r}{c}\right)}, \; \mathfrak{A}_y = 0, \; \mathfrak{A}_z = 0, \; \Omega = 0$$

and

$$\mathfrak{E}_x = \frac{\partial^2 \mathfrak{A}_x}{\partial x^2} + \frac{n^2}{c^2}\mathfrak{A}_x\,,$$

as may be easily seen, if the equations

$$p = \frac{1}{in}, \quad q = 1, \quad \dot{\mathfrak{A}}_x = in\,\mathfrak{A}_x$$

are taken into account.

In the differential coefficients of $\mathfrak{A}_x$ we may omit all terms containing the square and higher powers of $\frac{1}{r}$. Hence, in a point of the axis of z, which passes through the point P, $\frac{\partial^2 \mathfrak{A}_x}{\partial x^2} = 0$.

In this way, the electric force in the aether immediately before the plate is found to be

$$\mathfrak{E}_x = \frac{a_1 n^2 S}{4 \pi c^2 r} e^{in\left(t - \frac{r}{c}\right)}.$$

Its amplitude is

$$\frac{a_1 n^2 S}{4 \pi c^2 r} \quad \dots\dots\dots \quad (38)$$

and that of the current $\mathfrak{C}_x$ within the plate

$$\frac{a_1 n^2 S f}{4 \pi c^2 r}.$$

This must be equal to the expression (37). The solution of our problem is therefore

$$a_1 = \frac{4 \pi c}{n} \sqrt{\frac{2 k \alpha_1 dn}{S}} \quad \dots\dots \quad (39)$$

In the preceding formulae S means the volume of the portion of the plate we have considered. Now, after having decomposed this portion into a large number of elements of volume **s**, we may bring about just the same radiation by applying in each of these an electromotive force in the direction of OX with the amplitude

$$a_1 = \frac{4 \pi c}{n} \sqrt{\frac{2 k \alpha_1 dn}{\mathbf{s}}}, \quad \dots\dots \quad (40)$$

provided only we suppose the electromotive forces in all these elements **s** to be independent of each other, so that their phases are distributed at random over the elements.

Indeed, from the fact that the force whose amplitude is (39), acting in the space S, gives rise to a radiation represented by (36), we may conclude that an electromotive force with the amplitude (40), when applied to the element **s**, will produce a flow of energy

$$\frac{k \mathbf{s} \alpha_1 f^2 \omega' dn}{c r^2}$$

across the element ω'. A similar expression holds for each element **s** and, on account of the circumstance that the vibrations due to the separate elements have all possible phases, we may add to each other all these expressions. We are thus led back to the result contained in (36).

§ 14. Whatever be the nature of the processes in the interior of an element of volume, by which the radiation is caused, they can

undoubtedly be considered as determined by the state of the matter contained within the element; for this reason an electromotive force equivalent to those processes can only depend on quantities determined by that state; it cannot be altered by changing the state of the system outside the element considered, or the form and magnitude of the whole body. The formula (40), which indeed is determined by the state of things within the element **s**, must therefore be applied to an element of volume of all ponderable matter. It will be clear also that we have to add the following formulae for the amplitudes of the electromotive forces in the directions of y and z,

$$a_2 = \frac{4\pi c}{n} \sqrt{\frac{2k\alpha_2 dn}{\mathbf{s}}}, \quad a_3 = \frac{4\pi c}{n} \sqrt{\frac{2k\alpha_3 dn}{\mathbf{s}}} \quad \dots \quad (41)$$

As to the phases of the three electromotive forces, we shall suppose them not only to change irregularly from one element to another, but also to be mutually independent in one and the same element, so that the phase-differences between the three forces have very different values in neighbouring infinitely small spaces. In virtue of this assumption the intensities of the radiation due to the different causes may be added to each other.

Till now we have only accounted for the flow of energy (36), a part of the total flow represented by (35). We shall show in the next paragraph that the remaining part

$$\frac{k S \alpha_2 g^2 \omega' dn}{c r^2} \quad \dots\dots\dots\dots \quad (42)$$

is precisely the radiation brought about by the electromotive forces we have supposed to exist in the direction of OY, and that the forces acting in that of OZ cannot give rise to a radiation across the element ω'. After having proved these propositions, we may be sure that, as far as the electric vibrations parallel to OX are concerned, the plate has exactly the emissivity that is required by KIRCHHOFF's law. Of course, the same will be true for the vibrations in the direction of OY.

§ 15. It may be immediately inferred from the theorem of § 7, *a* that the electromotive forces applied to the plate in the direction of OZ, i. e. perpendicularly to the surfaces, cannot contribute anything to the radiation we have considered. Indeed, we know already that an electromotive force $\mathfrak{E}_{ex}$ existing in the aether at the point P can produce no current $\mathfrak{C}_z$ in the plate; consequently, an electromotive force $\mathfrak{E}_{ez}$ in the plate cannot cause a current $\mathfrak{C}_x$ at the point P.

As to the effect produced by the electromotive force with the amplitude a_2 acting in the direction of OY, this may be found by a reasoning similar to that we have used in §§ 12 and 13. Let us suppose for a moment an electromotive force of the same direction and intensity to exist in an element of volume **s** of the aether near the point P. The amplitude of the electric force $\mathfrak{E}_x$ in the aether immediately before the plate will then be (cfr. 38)

$$\frac{a_2 n^2 \mathbf{s}}{4\pi c^2 r},$$

and that of the current $\mathfrak{C}_y$ in the plate

$$\frac{a_2\, n^2\, \mathbf{s}\, g}{4\pi\, c^2\, r}.$$

If follows from this that, if the element **s** in the plate is the seat of an electromotive force $\mathfrak{E}_{ey}$, with amplitude a_2, the current $\mathfrak{C}_x = \dot{\mathfrak{E}}_x$ at the point P will have this same value. The amplitude of the electric force $\mathfrak{E}_x$ will be

$$b' = \frac{a_2\, n\, \mathbf{s}\, g}{4\pi\, c^2\, r} = \frac{g}{cr} \sqrt{2k\, \mathbf{s}\, a_2\, dn}$$

and the corresponding radiation across the element ω'

$$\frac{1}{2} c b'^2\, \omega' = \frac{k\, \mathbf{s}\, a_2\, g^2\, \omega'\, dn}{cr^2}.$$

This leads immediately to the expression (42).

§ 16. We are now in a position to form an idea of the state of radiation in a system of bodies of any kind. After having divided them into elements of volume **s**, and after having determined the principal directions at every point, we conceive in each element the electromotive forces whose amplitudes are determined by (40) and (41), the phases of all these forces being wholly independent of each other. In representing to ourselves the state of things obtained in this way, we must keep in mind:

1st. that the principal directions and the coefficients a_1, a_2, a_3 will, in general, change from point to point and will depend on the frequency n.

2ndly. that for each frequency n or rather for each interval dn of frequencies, we must assume electromotive forces of the intensity we have defined in what precedes, all these forces existing simultaneously.

We shall now show that, if the temperature is uniform throughout the system, the condition for the equilibrium of radiation will be fulfilled in virtue of our assumptions. Of course, it will suffice to prove this proposition for a single interval of frequencies dn.

Let **s** and **s'** be two elements of volume, arbitrarily chosen, h one of the principal directions of the first element, h' one of the principal directions of the other, a_h and $a'_{h'}$ the coefficients relating to these directions.

In virtue of the electromotive force $\mathfrak{E}_{eh}$ acting in **s** in the direction h, there will be in **s'** in the direction h' a current $\mathfrak{C}_{h'}$ with a certain amplitude $(\mathfrak{C}_{h'})$; by (31) the development of heat corresponding to this current will be per unit of time

$$\frac{1}{2} a'_{h'} (\mathfrak{C}_{h'})^2\, \mathbf{s}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Similarly, we may write

$$\frac{1}{2} a_h (\mathfrak{C}'_h)^2\, \mathbf{s} \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

for the heat developed in **s** on account of the current $\mathfrak{C}'_h$ produced in this element in the direction h by the electromotive force acting in **s'** in the direction h'.

Since each of the three electromotive forces in **s** calls forth a current in the element **s'** in each of its principal directions, there will be in all nine expressions of the form (43). These must be added to each other, as may be seen by observing that the total development of heat, represented by (31), is the sum of three parts, each belonging to one of the components of the current and that the three electromotive forces in **s** are mutually independent. The sum of the nine quantities will be the total amount of heat **s'** receives from **s**, and in the same way we must take together nine quantities of the form (44), if we wish to determine the amount of heat transferred from **s'** to **s**. We shall have proved the equality of the mutual radiations between the two elements, if we can show that for any two principal directions, the expressions (43) and (44) have the same value.

Let us call a_h and $a'_{h'}$ the amplitudes of the electromotive forces originating the currents whose thermal effects have been represented by (43) and (44). Then, in accordance with (40) and (41),

$$a_h = \frac{4\pi c}{n} \sqrt{\frac{2k\, a_h\, dn}{\mathbf{s}}},\quad a'_{h'} = \frac{4\pi c}{n} \sqrt{\frac{2k\, a'_{h'}\, dn}{\mathbf{s}'}}. \quad . \quad (45)$$

Now, by the general theorem of § 7, a the amplitudes $(\mathfrak{C}_{h'})$ and $(\mathfrak{C}'_h)$ in (43) and (44) are proportional to $a_h\mathbf{s}$ and $a'_{h'}\mathbf{s}'$. Taking into account the formula (45), we infer from this

$$(\mathfrak{C}_{h'})^2 : (\mathfrak{C}'_h)^2 = a_h^2\, \mathbf{s}^2 : a'^2_{h'}\, \mathbf{s}'^2 = a_h\, \mathbf{s} : a'_{h'}\, \mathbf{s}',$$

an equation, which leads directly to the equality of (43) and (44).

If the system of bodies is entirely shut off from its surroundings, the equality of the mutual radiation between any two elements implies that the state is stationary.

In order to show this, we fix our attention on one particular element **s**, denoting all other elements by **s'**. By what has been said, the sum w_1 of all quantities of heat which **s** receives from the elements **s'** will be equal to the sum w_2 of the quantities of heat it gives up to them. But, if the system is isolated from other bodies, each quantity of energy lost by **s** will be found back in one of the elements **s'**; w_2 is therefore the total amount of energy radiating from **s** and the equality $w_1 = w_2$ means that **s** gains as much heat as it loses.

§ 17. We shall finally assume that the system contains a certain space which is occupied by an isotropic and homogeneous body L, perfectly transparent to the rays; we shall examine the electromagnetic state existing in this medium, if all bodies are kept at the same temperature. To this effect, we must begin by a discussion of the radiation that would take place, if the body L extended to infinity, and if it were subjected to an electromotive or magnetomotive action (§ 7) at a certain point O.

A perfectly transparent body is characterized by the absence of all thermal effects. This means that the coefficient α is zero, as appears by (30). We have therefore

$$p = -i\beta, \quad \ldots \ldots \quad (46)$$

the coefficient q being real and positive, and the equation (17) becomes

$$v = c\sqrt{\beta q n}, \quad \ldots \ldots \quad (47)$$

I shall take here the positive value.

Let us first apply to an element of volume S at the point O, which I shall take as origin of coordinates, an electromotive force $\mathfrak{E}_{ex} = ae^{int}$, but no magnetomotive force. Then

$$\mathfrak{A}_x = \frac{a\,\mathrm{S}}{4\pi r} e^{in\left(t-\frac{r}{v}\right)},\ \mathfrak{A}_y = 0,\ \mathfrak{A}_z = 0,\ \mathfrak{Q} = 0.$$

What we want to know, is the amount of energy radiating from O, i. e. the flow of energy through a closed surface surrounding this point. In calculating this flow, the form and dimensions of the surface are indifferent; we shall therefore consider a sphere with O as centre and with an infinite radius r.

Then we may omit all terms in $\mathfrak{E}$ and $\mathfrak{H}$ containing the square and higher powers of $\frac{1}{r}$, and we find from (15) and (16), attending

to (46) and (47) and taking the real parts

$$\mathfrak{E}_x = \frac{a\,\mathrm{S}\,n^2}{4\pi r v^2} \cdot \frac{r^2 - x^2}{r^2} \cos n\left(t - \frac{r}{v}\right),$$

$$\mathfrak{E}_y = -\frac{a\mathrm{S}n^2}{4\pi r v^2} \cdot \frac{xy}{r^2} \cos n\left(t - \frac{r}{v}\right), \quad \mathfrak{E}_z = -\frac{a\mathrm{S}n^2}{4\pi r v^2} \cdot \frac{xz}{r^2} \cos n\left(t - \frac{r}{v}\right),$$

$$\mathfrak{H}_x = 0,\ \mathfrak{H}_y = \frac{a\mathrm{S}n}{4\pi r\beta c v}\frac{z}{r} \cos n\left(t - \frac{r}{v}\right),\ \mathfrak{H}_z = -\frac{a\mathrm{S}n}{4\pi r\beta c v}\frac{y}{r} \cos n\left(t - \frac{r}{v}\right).$$

The electric and magnetic force being known, the flow of energy through the sphere may be calculated by means of (23). Its value is

$$\frac{a^2\,\mathrm{S}^2\,n^3}{12\,\pi\,\beta\,v^3} \cdot \quad \ldots \ldots \quad (48)$$

If we perform a similar calculation in the assumption of a magnetomotive force with amplitude a, acting in the space S, the result is

$$\frac{a^2\,\mathrm{S}^2\,n^4}{12\,\pi\,q\,v^3} \cdot \quad \ldots \ldots \quad (49)$$

§ 18. Let P be a point of the body L mentioned at the beginning of the preceding paragraph, l an arbitrarily chosen direction and let us seek the amplitude $(\mathfrak{E}_l)$ of the electric current, or rather the square of the amplitude, produced by the radiating bodies, confining ourselves to the interval of frequencies dn.

We shall divide the bodies into elements of volume **s** and we shall denote, for one of these elements lying at the point Q, by h one of the principal directions, by α_h the coefficient relating to it, and by a_h (cfr. (45)) the amplitude of the electromotive force acting in that direction.

The amplitude $(\mathfrak{E}_l)$ produced by this force at the point P is equal to the amplitude of the current $\mathfrak{E}_h$, existing in the element **s**, if an electromotive force $\mathfrak{E}_{el}$, having the amplitude $\frac{a_h\,\mathbf{s}}{\mathrm{S}}$ is applied to an element of volume S of the aether near P. In order to express myself more briefly, I shall understand by A the radiation that would be excited by an electromotive action at the point P in the direction l of such intensity that the product $(\mathfrak{E}_{el})$ S has the value 1. The amplitude $(\mathfrak{E}_l)$ in P, of which we have just spoken, will be found if we multiply by $a_h\,\mathbf{s}$ the value which, in that state, $(\mathfrak{E}_h)$ would have in the element **s**. Hence

$$(\mathfrak{E}_{lP})^2 = a_h^2\,\mathbf{s}^2\,(\overset{\mathrm{A}}{\mathfrak{E}}_{hQ})^2 = \frac{32\,\pi^2\,c^2\,k\,\alpha_h\,\mathbf{s}\,dn\,(\overset{\mathrm{A}}{\mathfrak{E}}_{hQ})^2}{n^2}$$

$$= \frac{64\,\pi^2\,c^2\,k\,dn}{n^2}\overset{\mathrm{A}}{w}_h, \quad (50)$$

if we write w_h^A for the development of heat in the element **s**, which, in the state A, is due to the current in the principal direction h.

Now, starting from the expression (50), we shall obtain the total value of $(\mathfrak{E}_{lP})^2$ by an addition, in which all elements **s**, each with its three principal directions, must be taken into account. In a system, completely shut off from surrounding bodies, $\Sigma\, w_h^A$ will be the total amount of energy, emitted by P in the state A; we can therefore determine it by the formula (48), putting $a\,S = 1$. This leads to the result

$$(\mathfrak{E}_{lP})^2 = \frac{16\,\pi\,k\,c^2\,n\,dn}{3\,\beta\,v^3}.$$

In the same way, using the theorem of § 7, b and the expression (49), I find

$$(\mathfrak{H}_{lP})^2 = \frac{16\,\pi\,k\,c^2\,n^2\,dn}{3\,q\,v^3}.$$

These results being independent of the place of the point P and the choice of the direction l, we come to the conclusion that the state of things is the same in all parts of the medium L and that both the electric and the magnetic vibrations take place with equal intensities in all directions. The amount of the electric and magnetic energy per unit of volume is now easily found. According to § 4 the first is

$$\frac{1}{4}\,n\,\beta\,[(\mathfrak{D}_x)^2 + (\mathfrak{D}_y)^2 + (\mathfrak{D}_z)^2],$$

for the value of which one finds

$$\frac{4\,\pi\,k\,c^2\,dn}{v^3},$$

by remembering that for every direction l,

$$(\mathfrak{D}_l)^2 = \frac{1}{n^2}\,(\mathfrak{E}_l)^2.$$

The magnetic energy may likewise be determined. Referred to unit volume it has the value

$$\frac{1}{4}\,q\,[(\mathfrak{B}_x)^2 + (\mathfrak{B}_y)^2 + (\mathfrak{B}_z)^2],$$

and this is easily calculated, since for every direction l,

$$(\mathfrak{B}_l)^2 = \frac{1}{n^2}\,(\mathfrak{H}_l)^2.$$

The result is that the two kinds of energy are distributed over the body l with equal densities. This has been known for a long

time, as has also been the rule implied in our formulae, that these densities are inversely proportional to the cube of the velocity of propagation v. It must further be noticed that, if the medium L is aether, the density of the energy of the radiation becomes

$$\frac{8\,\pi\,k\,dn}{c}.$$

This agrees with the meaning we have originally attached to the coefficient k (§ 11).

§ 19. There is one point in the foregoing considerations that may at first sight seem strange, viz. that the intensity of the electromotive forces we have imagined should depend on the magnitude of the elements of volume **s**. It must be kept in mind however, that these forces have no real existence, and that we do not pretend to have found something concerning the causes by which the phenomena are produced. That the magnitude of the electromotive forces must be taken inversely proportional to the square root of the volume of **s** is simply a consequence of our assumption that the force has the same phase in all points of such an element. For a given amplitude of the electromotive force, the radiation would therefore be proportional to $\mathbf{s}^2$, and we had to make such assumptions concerning that amplitude, that the radiation became proportional to **s** itself.

In connection with these remarks it must be observed that we have no reasons for ascribing to the dimensions of the elements of volume some particular value. These dimensions are indifferent as long as we consider only the radiation at finite distances and the transfer of energy between neighbouring molecules lies outside the theory I have here developed.

Physiology. — *"On the ability of distinguishing intensities of tones".* By Prof. H. ZWAARDEMAKER. (Report of a research made by A. DEENIK.)

The "Unterschiedsschwelle" for impulsive sounds (dropping bullets and hammers) has been studied frequently and many-sidedly, but regarding the "Unterschiedsschwelle" for intensities of tone we have had at our disposal till now only some information communicated by M. WIEN in his thesis.

M. WIEN found the value of the "Unterschiedsschwelle" for the three tones, to which he limited his investigation to be as follows: for a average 22.5°/₀ (with 18.2 and 27 for extremes) for e' 17.6°/₀