instructed Dr. FRIEDMANN to prepare both by a method leading to pure products, in order to end the present uncertainty.

The α-aminopropionic acid (commercial alanine) was converted in the usual manner into the hydrochloric methyl ester, which melted at 158°. By the action of silver oxide or aqueous caustic soda, in presence of ether, the *free methyl ester* was prepared, which is very volatile, and under a pressure of 15 m.m. passes over between 38° and 41° as a colourless liquid (spec. gr. 13,5° = 1,0309), which however, after some days changes into a solid mass (alanine anhydride), presumably owing to the interaction of the two functions — amine and organic ester. A nitrogen determination in the liquid gave figures corresponding with those required by the amino ester $CH_3.CHNH_2.CO_2CH_3$. The amino ester was mixed with a saturated methylalcoholic ammonia and left to itself for a few days. A distillation at 35° at a pressure of 20 mm. removed the ammonia and alcohol and left as residue a colourless oily liquid of a strongly alkaline reaction which is but little soluble in ether and benzene and solidifies in a dessiccator. After being recrystallised from alcohol, the substance on analysis appeared to be pure. It crystallises in needles, is very soluble in alcohol, very hygroscopic and melts at 62°. On prolonged heating ammonia is set free and alanine anhydride is formed, as was indeed to be expected. The *α-aminopropionic amide* $CH_3\,CH\,NH_2\,.\,CO\,.\,NH_2$ gives also a well-crystallised compound with hydrogen chloride, a fine crystallised orange-red chloroplatinate, which is readily soluble in water, but little so in alcohol, and a bright yellow picrate, little soluble in water, which on being recrystallised from alcohol melts at 199°. Thus we have sufficiently characterized this amide for the present. It seems to be decomposed already at ordinary temperature in an exsiccator.

The β-aminopropionic acid was prepared but with a slight modification, according to HOOGEWERFF and VAN DORP from succinimide [1]. Like the α-compound it was converted into the hydrochloric methyl ester, which melts at 95°.

From this was prepared in the manner described above the free *β-aminopropionic methyl ester*, which under 18 m.m. pressure distills at 57°—59° as a colourless liquid and is pure as follows from analysis. After a few hours it is decomposed with formation of crystals. By direct treatment with methylalcoholic ammonia the amide was at first obtained as an oily liquid, which was purified by repeated solution in methyl alcohol and precipitation with ether, when it gave good analytical results. On being cooled, it became solid, and on inoculation with a trace of the solid material, it yielded beautiful crystals, melting at 41°. The *β-aminopropionic amide* $NH_2.CH_2.CH_2.CO.NH_2$ is very hygroscopic, very soluble in alcohol, with difficulty in ether, has a strongly alkaline reaction, absorbs carbon dioxide from the atmosphere and yields a well-crystallised hydrochloride.

Both aminopropionic amides are, therefore, now known and not identical with the substance obtained by BAUMSTARK from urine.

[1]) This was simply prepared like many other amides (when they and their acids can resist a fairly high temperature) by heating the acid in a current of ammonia until no more water is expelled.

Physics. — *"Remarks concerning the dynamics of the electron."* By Prof. J. D. VAN DER WAALS JR. (Communicated by Prof. J. D. VAN DER WAALS).

The theory of electrons is usually deduced from the following equations:

$$div\ \mathfrak{d} = \varrho \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (I)$$

$$rot\ \mathfrak{h} = \frac{1}{c}\,(\dot{\mathfrak{d}} + \varrho\,\mathfrak{v}) \quad . \ . \ . \ . \ . \ . \ . \quad (II)$$

$$rot\ \mathfrak{d} = -\frac{1}{c}\,\dot{\mathfrak{h}} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (III)$$

$$div\ \mathfrak{h} = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (IV)$$

$$\mathfrak{f} = \mathfrak{d} + \frac{1}{c}\,[\mathfrak{v}\,\mathfrak{h}] \quad . \ . \ . \ . \ . \ . \ . \quad (V)$$

The units and notations used are those of LORENTZ's article on the "Elektronentheorie" in the "Encyclopädie der Mathematischen Wissenschaften" V 14.

The equations (I) . . . (IV) determine the field, the motions of the electrons being given. Equation (V) will be independent of the former four, and determine the motion of the electron in the electromagnetic field, $\varrho\mathfrak{f}$ representing the force exercised by the field on the electric charge. The application of equation (V) is however attended by peculiar difficulties. If in mechanics a body is given its mass is also supposed to be known; then if we know the force, the acceleration, and the law according to which it will move may be calculated. Now if in forming $\mathfrak{f}$ we take into account all the forces also those excited by the electron itself, then the case that we ascribe a "real" or "material" mass to the electron offers no fundamental difficulty. For the case, however, in which the electron has no real mass the equation (V) assumes the form:

$$\text{force} = 0$$

without a term with $\frac{d\mathfrak{v}}{dt}$ in the righthand member which enables us to determine the acceleration. If on the other hand in forming $\mathfrak{f}$ we take into account only the external forces, we may introduce the electromagnetic mass m' and write:

$$\text{force} = m' \frac{d\mathfrak{v}}{dt}.$$

The mass m', however, is not known, and we can only calculate it if we know already the law according to which the electron moves. But then of course we need not make use of equation (V) any more.

ABRAHAM [1]), accordingly, does not make any use of equation (V) for determining the motion of the electron. He does use an equation which in LORENTZ's notation, the integrals being taken throughout infinite space, may be written as follows:

$$\iiint \varrho \left(\mathfrak{d} + \frac{1}{c} [\mathfrak{v}\, \mathfrak{h}]\right) dS = \frac{1}{c} \iiint \frac{\partial\, [\mathfrak{d}\, \mathfrak{h}]}{\partial t} dS.$$

But this equation is deduced from the equations $(I) \ldots (IV)$ and so it is not equivalent to equation (V) which must be independent of them. The only use which is made of equation (V) is the introduction of the name force for the quantity $\varrho \left(\mathfrak{d} + \frac{1}{c} [\mathfrak{v}\, \mathfrak{h}]\right)$, of the name momentum for $\frac{1}{c} [\mathfrak{d}\, \mathfrak{h}]$, and of the name electric mass for the quotient of the so defined force and the acceleration. But the real problem: how will an electron with given shape and charge move in a given field, must be solved beforehand independently of this nomenclature.

Yet it is evident that the equations $(I) \ldots (IV)$ will in general be insufficient for the determination of the motion of an electrical system. In the case that we ascribe a real mass to the electron, it is obvious that we must know the force acting on it. But also on the supposition that the electron has no real mass — and in what follows we will confine ourselves to this case — another set of equations is required for the determination of the motion. For equation (II) enables us to determine $\dot{\mathfrak{d}}$ if $\mathfrak{v}$ is always known, and inversely to determine $\mathfrak{v}$ if we know $\dot{\mathfrak{d}}$, but it does not enable us to determine both these quantities. The assumption that the motion is quasi stationary is equivalent to a relation between $\mathfrak{v}$ and $\dot{\mathfrak{d}}$.

[1]) ABRAHAM, "Dynamik des Elektrons." Ann. der Physik IV. B. 13, 1904, bl. 105.

For such a motion the equations $(I) \ldots (IV)$ are therefore sufficient to determine the motion. If the motion is not quasi stationary then the equations $(I) \ldots (IV)$ are not sufficient, and we must make use of equation (V), which may be written:

$$\iiint \varrho \left\{ \mathfrak{d} + \frac{1}{c} [(\mathfrak{v} + [\mathfrak{g}\, \mathfrak{r}])\, \mathfrak{h}] \right\} dS = 0 \quad . \quad . \quad (Va)$$

the integral being taken throughout the electron.

If we wish to state the meaning of this formula with the aid of the conceptions force and mass, we may say: the real mass of the electron being zero, it is impossible that a force should act on it. We may, however, set these conceptions aside, and simply state: the electron places itself and moves in the electric field in such a way, that the relation (V^a) is permanently satisfied.

It is true, this equation has the form: force $= 0$ without $m \frac{d\mathfrak{v}}{dt}$ in the righthand member. Yet it may serve to determine the motion. This is owing to the fact that the expression of the force itself contains the velocity $\mathfrak{v}$ and the angular velocity $\mathfrak{g}$. In general we may choose such values for these quantities that the equation (V^a) is satisfied. To some extent therefore we return with the dynamics of an electron to the standpoint of mechanics before GALILEI: the forces do not determine the acceleration but the velocity. If we might assume $\mathfrak{d}$ and $\mathfrak{h}$ to be given throughout all space and at all times, $\mathfrak{d}$ and $\mathfrak{g}$ would be determined by the place of the electron, and we should get a differential equation of the first order for the determination of the motion of the electron.

The question is in reality less simple, because $\mathfrak{d}$ and $\mathfrak{h}$ depend on the former motion of the electron. This causes a time-integral of a function of $\mathfrak{v}$ and $\mathfrak{g}$ to occur in the equation of motion of the electron. So we get integral equations as SOMMERFELD has used in his treatises "Zur Elektronentheorie I, II and III". [1]) In some cases the integrations may be effected, and then we get functional equations.

If the electron moves rectilinearly without rotation, and if it moreover has an axis of symmetry the direction of which coincides with the direction of the translation, then the terms of equation (Va) which contain $\mathfrak{v}$ or $\mathfrak{g}$ disappear and the equation reduces to:

$$\iiint \varrho\, \mathfrak{d}\, dS = 0\,.$$

In this case it is no longer possible to satisfy the equation by

[1]) Göttinger Nachrichten 1904, p. 99 and 363 and 1905 p. 201.

means of a suitable choice of the value of $\mathfrak{v}$ and $\mathfrak{g}$, and now it is the *place* of the electron which must be such that the equation is satisfied. If the electron stood still the equation would cease to be satisfied in a following moment because of the propagation of the field-forces, it must therefore suffer a displacement in such a way that the relation *continues* to be satisfied. So the equation determines the velocity, though the velocity itself does not occur in it.

This remark may perhaps serve to elucidate the results of SOMMERFELD concerning the motion with a velocity greater than that of light, and this is principally my aim with this communication. In the following I shall denote a velocity, greater than that of light, with $\mathfrak{V}$ and one smaller with $\mathfrak{v}$.

We see at once that the supposition of SOMMERFELD that the velocity of an electron moving with $\mathfrak{V}$ will suddenly decrease to $\mathfrak{v}$ when the external force is suddenly suppressed, cannot be accurate. For if we take $\mathfrak{d}$ to be the sum of two parts $\mathfrak{d}_1$ the external field and $\mathfrak{d}_2$ the field of the electron itself, then we have at the moment t before the suppression of the external field:

$$\iiint \varrho\, (\mathfrak{d}_1 + \mathfrak{d}_2)\, dS = 0.$$

But as $\mathfrak{v}$ requires an external force, $\iiint \varrho\, \mathfrak{d}_1\, dS$ is not zero, so neither can $\iiint \varrho\, \dot{\mathfrak{d}}_2\, dS$ be zero. This last quantity is independent of the velocity at the moment t itself, and so it cannot be made to disappear by any choice of the velocity, and there is no possible way in which equation (Va) can be satisfied.

If we imagine the velocity of an electron moving with $\mathfrak{V}$ momentarily to decrease to $\mathfrak{v}$, then the required external force will not suddenly become zero, but at the first instant it remains unchanged, and only gradually it varies in accordance with the new mode of motion. This thesis applies to every discontinuity in the velocity provided the motion be rectilinear and the electron have the required symmetry. For the case that the initial velocity is zero it follows from SOMMERFELD's complete calculation of the force. We see again the conformableness of the dynamics of an electron with a theory of mechanics in which no inertia is assumed: the force required for a discontinuous change in the velocity is not only not infinite, but even zero; the force, which acts before the discontinuity, remains unchanged at the moment of the discontinuity.

We cannot be astonished at the fact that we do not find a possible

way of motion for an electron moving with $\mathfrak{V}$, when the external force is suddenly suppressed. The same applies to an electron moving with $\mathfrak{v}$; if the motion is accelerated, and so if a force acts on the electron, and if this force is suddenly suppressed, the equation (Va) cannot be satisfied in any way. This is because the momentary disappearance of the external force is an impossible supposition. Even an infinite acceleration would not satisfy equation (Va). The internal force namely depends only on the former motion of the electron, and not on the velocity or the acceleration at the moment itself. SOMMERFELD's conclusion that a motion with $\frac{\partial \mathfrak{v}}{\partial t} = \infty$ does not require an external force holds only if the initial velocity is $\mathfrak{v}$, and is nothing else but a statement in other words of the fact that the force acting on an electron whose velocity is at the moment t momentarily — i. e, with infinite acceleration — brought from $\mathfrak{v}$ to $\mathfrak{V}$, is zero at the moment t. If however we begin with a constant velocity $\mathfrak{V}_1$, and change the velocity at the moment t suddenly to $\mathfrak{V}_2$, then the force is not zero at the moment t, though the acceleration be infinite, but it has that value which corresponds with a constant velocity $\mathfrak{V}_1$.

It may however be asked what will happen, if the force acting on an electron with $\mathfrak{V}$ does not suddenly decrease to zero, but gradually. SOMMERFELD says about this case only that the sudden fall to $\mathfrak{v}$, which he expects from a sudden suppression of the force, will make room for a gradual fall. But as his expectation concerning the case of a sudden suppression of the force appeared to be inaccurate, we might suppose that also this expectation will appear not to be satisfied. The more so because SOMMERFELD found a negative value for the electric mass of an electron moving with $\mathfrak{V}$. We might therefore expect that a decrease of the force would cause an acceleration. This, however, is not the case, and here we see how risky it is to introduce the conception of mass in the theory of the motion of electrons, to which it is essentially strange.

The negative mass, which SOMMERFELD ascribes to the electron means nothing else, but that in order to move with a given $\mathfrak{V}_1$ the electron requires a greater force when in the active interval the velocity was on an average greater than $\mathfrak{V}_1$, a smaller force when it was less. By active interval is meant the time during which the electron emitted the fieldforces, which at the moment t act on the electron. The greater the velocity during the active interval, the greater the force, and inversely the smaller the velocity the smaller the force[1]). But it does not follow from this that also a greater force is required if the retardation exists only in the future. On the contrary when the velocity has decreased to $\mathfrak{V}_2 < \mathfrak{V}_1$ the velocity during the active interval has been smaller than $\mathfrak{V}_1$ on an average, and so also the force required will be smaller than that which corresponds to a constant velocity $\mathfrak{V}_1$. So with a gradual decrease of the velocity corresponds a gradual decrease of the force. The reverse of this thesis is not always true: if $\frac{d\mathfrak{F}}{dt}$ is a continuous function of t then the velocity will also vary continuously. If on the other hand $\frac{d\mathfrak{F}}{dt}$ is discontinuous though $\mathfrak{F}$ be continuous then $\mathfrak{v}$ will vary discontinuously.

A diminution of the force is therefore accompanied by a diminution of the velocity, and inversely. The behaviour of an electron moving with $\mathfrak{V}$ corresponds in this respect with that of a body with a positive mass. If the force acting on the electron decreases gradually to zero, the velocity will fall to $\mathfrak{v}$.

Though it seems to me that there is no reason to doubt whether the behaviour of an electron has been described here accurately though only in general outlines, and though a complete calculation of the motion is not practicable in consequence of the great intricacy of the formulae, I will show in one simple case that the force required for a given motion agrees with the above description. I imagine to that purpose an electron which for some time moves with a constant velocity $\mathfrak{V}$. At the instant t the motion is suddenly accelerated with a constant acceleration p. In order to render the calculation possible we will assume that we may apply the formulae for quasi stationary motion. We will calculate the force at an instant t in the first interval[2]), so $t > \tau'$. The calculation does not present any difficulties, and can be carried out in the way indicated by SOMMERFELD. After introduction of the approximation for the quasi stationary motion we may everywhere separate the terms as they would be for a constant velocity $\mathfrak{V}$, (we call the sum of these terms $\mathfrak{F}_1$) and the supplementary terms which depend on the acceleration, and whose sum will be denoted by $\mathfrak{F}_2$. In this way we find:

[1]) This rule is given by SOMMERFELD though his calculations show that it does not hold good with perfect generality. In most cases and also in the present one it will give a true idea in general outlines of the value which the force must assume.

[2]) SOMMERFELD III p. 206.

$$
-\frac{32\pi a^4}{3\varepsilon^2}\mathfrak{F} = -\frac{32\pi a^4}{3\varepsilon^2}\mathfrak{F}_1 - \frac{8}{5}\frac{ac^2}{v^2}\frac{pt^2}{4}
$$

$$
+ p\frac{c}{2v^2(v+c)}\int_0^{x_1}\left(2\varphi + x\frac{d\varphi}{dx}\right)x\,dx - p\frac{c}{v^3}\int_0^{x_1}\varphi x\,dx
$$

$$
- pt\frac{c}{v^2}\int_0^{x_1}\left(2\varphi + x\frac{d\varphi}{dx}\right)dx + pt\frac{c(v-2c)}{v^3}\int_0^{x_1}\varphi\,dx
$$

$$
- pt^2\frac{c(v+c)}{2v^2}\int_{x_1}^{2a}\left(2\varphi + x\frac{d\varphi}{dx}\right)\frac{1}{x}\,dx + pt^2\frac{c^2(v+c)}{v^3}\int_{x_1}^{2a}\varphi\frac{1}{x}\,dx
$$

+ six other integrals which are obtained by substituting $-c$ for c and x_2 for x_1 in the above.

The signification of the symbols is as follows: a is the radius of the spherical electron which is supposed to be charged with homogeneous cubic density; ε is the charge of the electron, c the velocity of light, v the numeric value of $\mathfrak{V}$; φ the function $2a - x + \frac{1}{20}\frac{x^3}{a^2}$; x_1 is $(v+c)t + {}^1/_2\,pt^2$ and $x_2 = (v-c)t + {}^1/_2\,pt^2$. In the expressions for x_1 and x_2 the term ${}^1/_2\,pt^2$ may, however, be neglected.

Without performing the integrations completely we may draw the following conclusions:

1st. All the terms of $\mathfrak{F}_2$ contain t as a factor. So we have $\mathfrak{F} = \mathfrak{F}_1$ if t vanishes. No sudden increase of the force is therefore required if the motion is suddenly accelerated, as is the case with a body with positive mass; neither a sudden diminution of the force as would be the case with a body with negative mass. The force remains unchanged.

2nd. No terms with the first power of t occur in $\mathfrak{F}_2$, therefore $\frac{d\mathfrak{F}}{dt_{(t=0)}} = 0$. Even the derivative $\frac{d\mathfrak{F}}{dt}$ is therefore continuous at the point $t = 0$. This agrees with our remark that a discontinuity in the derivative of $\mathfrak{F}$ only occurs if the velocity changes discontinuously.

3rd. For establishing the sign of $\mathfrak{F}_2$ for $t =$ very small, we have only to take into account the terms with t^2. There are also terms with $t^2\,l(t)$ but the sum of their coefficients is zero. If we perform the integrations as far as is required we find:

$$
\left\{+\frac{1}{3}\frac{c^2}{v^2} + \frac{c^2(v^2-c^2)}{v^3}\,l\,\frac{v+c}{v-c}\right\}2\,a\,p\,t^2 .
$$

$+ \mathfrak{F}$ representing the force of the field of the electron itself, $- \mathfrak{F}$ is the external field required for the motion. So we see that the sign of the external force $- \mathfrak{F}_2$ agrees with that of p, and that therefore acceleration requires increase, retardation decrease of the external force.

We conclude that the behaviour of an electron moving with $\mathfrak{V}$, though in many respects it differs considerably from that of an ordinary body, does not show at all that paradoxal character, to which we should conclude from the expression *negative mass*. Nothing prevents us from assuming that electrons really can behave in such a way. Accordingly I do not see any reason for assuming with WIEN [1]) that a moving electron must suffer a deformation in order that the possibility of a motion with $\mathfrak{V}$, as it requires an infinite amount of energy, will be precluded.

Finally a remark concerning the series of the emission spectra of elements. The equations of motion of the electron are integral or functional equations, and may be developed into differential equations of an infinitely high order. An infinite number of constants occur accordingly in the solution. If the equations are linear, these constants represent the amplitudes and phases of harmonic vibrations; the system may therefore vibrate with an infinite number of periods [2]). We are inclined to think that the periods of the lines of a spectral series are the solutions of such an equation. We have then the great advantage that we need not ascribe to the electron a degree of freedom for each line in the spectrum. A degree of freedom in the atom is then not required for each line, but only for each series of lines.

SOMMERFELD tries to account for the spectral series by means of the vibrations an electron performs when it is not subjected to external forces. The periods which he finds, do not agree with those of light. It seems to me that we might have expected this a priori. For the vibrations of light are not emitted by isolated electrons but they are characteristic for atoms or positive ions, and are influenced by the forces by which the electron is connected to the other parts of the atoms or ions. But also with the aid of these forces we cannot account for the spectral series without a much better insight into

[1]) W. WIEN. Über Elektronen. Vortrag gehalten auf der 77. Versammlung Deutscher Naturforscher und Ärzte in Meran p. 20.

[2]) Comp. also these Proceedings March 1900 p. 534. Then however, I thought erroneously that the solution obtained in this way was different from that, which I had first developed with the aid of integrals of FOURIER.

the way in which these forces act, and of the properties of the electron, than we have as yet obtained. If e. g. we introduce the so called quasi-elastic force into the equations of motion of the electron, then this does not bring us any nearer to our aim. In order to show this we may write the equations for translation of an electron in the form of a differential equation as LORENTZ has done in equation 73 p. 190 of his article "Elektronentheorie" in the Encycl. der Math. Wiss. V 14. If we introduce the quasi-elastic force $-fx$ we may write the equation as follows:

$$fx + A_1 \frac{d^2x}{dt^2} + A_2 \frac{d^3x}{dt^3} + A_3 \frac{d^4x}{dt^4} + \ldots = 0$$

As it is only my aim to determine the order of magnitude I have not determined the coefficients (A_1 and A_2 have been determined by LORENTZ). The only thing we have to know is that the order of magnitude of the ratios of two successive coefficients is $\frac{A_{n+1}}{A_n} = \frac{a}{c}$. The solution of this equation is $x = \Sigma' c e^{st}$ where s is a root of the equation:

$$f + A_1 s^2 + A_2 s^3 + A_3 s^4 \ldots = 0$$

This equation has two kinds of roots, namely 1st two roots for which the other terms are small compared with $f + A_1 s^2$; these will represent the light vibrations; 2nd an infinite number of roots for which s is so large that f may be neglected compared with the other terms. For these s must be of the order $\frac{c}{a}$, and the period of the order $\frac{a}{c}$. The appearance of the term f has little influence on the value of these roots, the periods of these vibrations are therefore nearly independent of the quasi-elastic force, and an isolated electron might have executed vibrations with nearly the same periods. We might have expected a priori that we should find periods of the order $\frac{2a}{c}$: it represents the time required for the propagation of an electric force over the diameter of the electron. The periods of these vibrations are of the same order as those of the rotatory vibrations the periods of which have been accurately calculated in the interesting treatises of HERGLOTZ [1]) and SOMMERFELD.

The lines of the spectral series are not accounted for in this way. Yet the periods of the rotation and translation vibrations of the isolated electron must have a physical interpretation. Perhaps we should see them appear if we succeeded in forming the spectrum of RÖNTGEN radiation.

[1]) HERGLOTZ, Gött. Nachr., 1903.