Physics. — "*Derivation of the fundamental equations of metallic reflection from* CAUCHY's *theory*". By Prof. R. SISSINGH. (Communicated by Prof. H. A. LORENTZ).

1. It has been pointed out in a previous paper[1]) that the theories of metallic reflection drawn up by CAUCHY, KETTELER and VOIGT and that by LORENTZ lead to identical results. It must therefore also be possible in the theory of CAUCHY to derive the two relations which the three last theories furnish between index of refraction and coefficient of absorption for normal and oblique incidence of the light that penetrates into a metal, the so-called fundamental equations. These fundamental equations may first be obtained by paying regard to the connection of the quantities which the theory of CAUCHY and the other theories introduce for the description of the phenomenon. CAUCHY determines the so-called complex angle of refraction r by $\sin r = \sin i : \sigma e^{\iota\tau}$ and $\cos r = \varrho e^{\iota\omega}$ [2]). From this follows $1 - \frac{\sin^2 i}{\sigma^2 e^{2\iota\tau}} = \varrho^2 e^{2\iota\omega}$, so that:

$$\sigma^2 \cos 2\tau = \varrho^2 \sigma^2 \cos 2(\tau + \omega) + \sin^2 i \quad . \quad . \quad . \quad . \quad (1)$$

$$\sigma^2 \sin 2\tau = \varrho^2 \sigma^2 \sin 2(\tau + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we pay regard to the relations between σ, τ and n_0 and k_0, index of refraction and coefficient of absorption for normal incidence, and to the equations (17) and (18) of the preceding paper [2]), the equations (1) and (2) appear to be nothing but the fundamental equations, given in equation (6) and (7) of the previous paper.

2. On account of the close connection between the theories of metallic reflection it must, however, be also possible, to derive these fundamental equations from CAUCHY's theory without paying attention to the connection with the others. The fundamental idea of CAUCHY's theory is the introduction of a complex index of refraction. Denote this again by $n_0 + \iota k_0 = \sigma e^{\iota\tau}$, so that

$$n_0 = \sigma \cos \tau \quad , \quad k_0 = \sigma \sin \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\sin r = \sin i : \sigma e^{\iota\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

while we put

$$\cos r = \varrho e^{\iota\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let the XZ-plane of a rectangular system of coordinates be the plane of incidence of the light penetrating into the metal, and the YZ-plane the bounding plane of the metal, the X-axis being directed

[1]) SISSINGH, These Proc. VIII p. 377.

[2]) Loc. cit. p. 385.

from the surrounding medium to the metal. Assume plane waves to fall on the metal. The vector of light in the refracted ray is then determined by:

$$A \sin 2\pi \left\{ \frac{t}{T} - \frac{x \cos r + z \sin r}{\lambda} (n_0 + \iota k_0) \right\} \quad . \quad . \quad . \quad . \quad (6)$$

In this λ is the wave-length in the air [1]). The phase is determined with respect to a point in the bounding plane.

With the aid of (4) and (5) equation (6) passes into

$$A \sin 2\pi \left\{ \frac{t}{T} - \frac{x \varrho e^{\iota\omega} + z \sin i \, e^{-\iota\tau} : \sigma}{\lambda} (n_0 + \iota k_0) \right\} \quad . \quad . \quad . \quad (7)$$

(7) satisfies also the differential equations for the vector of light in the metal which are supposed homogeneous and linear, if the sine is replaced by a cosine.

If the arc occurring in (7) is called φ, also

$$A \cos \varphi - \iota A \sin \varphi \;^{2})$$

satisfies.

The light-vector in the metal can therefore be represented by

$$A e^{-2\pi a} \times e^{-2\pi\iota\left(\frac{t}{T} - b\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In this:

$$a = \left(\varrho x \sin \omega - z \sin i \frac{\sin \tau}{\sigma}\right) \frac{n_0}{\lambda} + \left(\varrho x \cos \omega + z \sin i \frac{\cos \tau}{\sigma}\right) \frac{k_0}{\lambda}. \quad (9)$$

$$b = \left(\varrho x \cos \omega + z \sin i \frac{\cos \tau}{\sigma}\right) \frac{n_0}{\lambda} - \left(\varrho x \sin \omega - z \sin i \frac{\sin \tau}{\sigma}\right) \frac{k_0}{\lambda}. \quad (10)$$

3. From (8) follows, that the planes of equal amplitude are represented by:

$$a = p_1 x + q_1 z = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In this is, according to (9)

$$q_1 = - \sin i \frac{\sin \tau}{\sigma} \frac{n_0}{\lambda} + \sin i \frac{\cos \tau}{\sigma} \frac{k_0}{\lambda}.$$

As from (3) follows

$$n_0 : k_0 = \cot \tau \quad , \quad \text{we have } q_1 = 0.$$

[1]) LORENTZ showed that, also when a complex index of refraction is introduced, at the bounding plane the values of the light vector in the two media harmonize. Cf. Theorie der terugkaatsing en breking, 1876, p. 160.

[2]) It appears from § 5 of the previous paper (loc. cit. p. 381), that if the index of refraction is put $n_0 \pm \iota k_0$, this expression is $A \cos \varphi \mp \iota A \sin \varphi$.

and the planes of equal amplitude run parallel to the bounding plane. This is necessary as it is assumed that the light enters the metal from the outside.

The planes of equal phase are represented by:

$$b = p_2 x + q_2 z = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we introduce again $n_0 : k_0 = cot\, \tau$, then according to (10)

$$p_2 = \frac{\varrho}{\lambda}\, k_0\, \frac{cos(\tau+\omega)}{sin\, \tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$q_2 = \frac{k_0}{\sigma\lambda} \frac{sin\, i}{sin \tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

4. Let α be the angle between the normals of the planes of equal amplitude and phase. The former running parallel to the bounding plane or the YZ-plane, α is the angle of the normal of the planes of equal phase with the X-axis. Thus $cos\, \alpha = p_2 : \sqrt{p_2{}^2 + q_2{}^2}$ or if we introduce the values p_2 and q_2 from (13) and (14):

$$cos\, \alpha = \varrho\, cos\, (\tau + \omega) : \sqrt{\varrho^2\, cos^2\, (\tau + \omega) + \frac{sin^2 i}{\sigma^2}} \quad . \quad . \quad (15)$$

From this follows:

$$sin\, \alpha = \frac{sin^2 i}{\sigma^2} : \left[\varrho^2\, cos^2\, (\tau + \omega) + \frac{sin^2 i}{\sigma^2}\right] \quad . \quad . \quad . \quad . \quad (16)$$

α being the angle of refraction corresponding to plane waves with an angle of incidence i (see § 2 of the preceding paper), we get:

$$n^2 = sin^2\, i : sin^2\, \alpha = \sigma^2\, \varrho^2\, cos^2\, (\omega + \tau) + sin^2\, i \; . \quad . \quad . \quad (17)$$

Let the coefficient of absorption belonging to n be k. Normal to the planes of equal amplitude the amplitude decreases over a distance x in ratio 1 to $e^{-2\pi k x : \lambda}$. As $q_1 = 0$, we get according to (8) and (9):

$$\frac{2\pi k x}{\lambda} = \frac{2\pi \varrho x}{\lambda} (n_0\, sin\, \omega + k_0\, cos\, \omega)$$

from which again follows, when $cot\, \tau$ is substituted for $n_0 : k_0$:

$$k = k_0\, \varrho\, sin\, (\tau + \omega) : sin\, \tau$$

or on account of (3):

$$k = \sigma\, \varrho\, sin\, (\tau + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

5. The fundamental equations follow immediately from the values found for the index of refraction and the coefficient of absorption. The equations (17) and (18) lead immediately to:

$$n^2 - k^2 = \sigma^2\, \varrho^2\, cos\, 2\, (\tau + \omega) + sin^2\, i\, .$$

According to (1) the second member of this equation is equal to $\sigma^2\, cos\, 2\tau$ or according to (3) to $n_0{}^2 - k_0{}^2$. In this way the first fundamental equation is obtained.

Further follows from (15), (17) and (18):

$$n\, k\, cos\, \alpha = \frac{1}{2}\, \sigma^2\, \varrho^2\, sin\, 2\, (\tau + \omega).$$

According to (2) the second member is equal to $\sigma^2\, sin\, 2\tau$ and so according to (3) to $n_0\, k_0$, and thus the second equation has also been derived.

To conclude we may remark, that here the reversed course has been taken from that by which in the preceding paper the occurrence of the so-called complex index of refraction was derived from the two fundamental equations [1]).

Mathematics. — *"A tortuous surface of order six and of genus zero in space Sp_4 of four dimensions."* By Prof. P. H. Schoute.

1. We begin by putting the following question:

"In space Sp_4 are given three planes $\alpha_1, \alpha_2, \alpha_3$ and in these are "assumed three projectively related pencils of rays. We demand the "locus of the common transversal of the triplets of rays corresponding "to each other."

Notation. We indicate the vertices of the rays of pencils by O_1, O_2, O_3, three corresponding rays and their transversal by l_1, l_2, l_3 and l, the points of intersection of l and l_1, l_2, l_3 by S_1, S_2, S_3 and the pencils of rays by $(l_1), (l_2), (l_3)$. Let further P_{23}, P_{13}, P_{12} indicate the points of intersection of the planes $\alpha_1, \alpha_2, \alpha_3$ two by two, and α the plane $P_{23}\, P_{13}\, P_{12}$ which has a line in common with each of the planes $\alpha_1, \alpha_2, \alpha_3$, namely with α_1 the line $P_{13}\, P_{12} = a_1$, with α_2 the line $P_{12}\, P_{23} = a_2$, with α_3 the line $P_{23}\, P_{13} = a_3$. We take for granted that not one of the three vertices O_1, O_2, O_3 coincides with one of the points P_{23}, P_{13}, P_{12}.

2. The answering of the given question offers no more difficulties, as soon as the locus of point S_1 in α_1 is known; so we shall first find this. Each ray l_1 of pencil (l_1) furnishing a single point S_1, it is a rational curve, whose degree surpasses the number of times a transversal l passes through O_1 with unity. Now two transversals l pass through O_1. For the pencil of planes $(O_1\, l_2)$ with $(O_1\, \alpha_2)$ as bearing space and $O_1\, O_2$ as axis marks on the line of intersection m of $(O_1\, \alpha_2)$ with α_3 a series of points (P) projectively related to the pencil of rays (l_3), from which ensues that there are two rays l_3 passing through their corresponding point P and that therefore there

[1]) See loc. cit. § 5.