and the planes of equal amplitude run parallel to the bounding plane. This is necessary as it is assumed that the light enters the metal from the outside.

The planes of equal phase are represented by:

$$b = p_2 x + q_2 z = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If we introduce again $n_0 : k_0 = cot\,\tau$, then according to (10)

$$p_2 = \frac{\varrho}{\lambda} k_0 \frac{cos(\tau + \omega)}{sin\,\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$q_2 = \frac{k_0}{\sigma\lambda} \frac{sin\,i}{sin\,\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

4. Let α be the angle between the normals of the planes of equal amplitude and phase. The former running parallel to the bounding plane or the YZ-plane, α is the angle of the normal of the planes of equal phase with the X-axis. Thus $cos\,\alpha = p_2 : \sqrt{p_2^2 + q_2^2}$ or if we introduce the values p_2 and q_2 from (13) and (14):

$$cos\,\alpha = \varrho\, cos\,(\tau + \omega) : \sqrt{\varrho^2\, cos^2\,(\tau + \omega) + \frac{sin^2 i}{\sigma^2}} \quad . \quad . \quad (15)$$

From this follows:

$$sin\,\alpha = \frac{sin^2 i}{\sigma^2} : \left[\varrho^2\, cos^2\,(\tau + \omega) + \frac{sin^2 i}{\sigma^2}\right] \quad . \quad . \quad . \quad . \quad (16)$$

α being the angle of refraction corresponding to plane waves with an angle of incidence i (see § 2 of the preceding paper), we get:

$$n^2 = sin^2\, i : sin^2\, \alpha = \sigma^2\, \varrho^2\, cos^2\,(\omega + \tau) + sin^2\, i \, . \quad . \quad . \quad (17)$$

Let the coefficient of absorption belonging to n be k. Normal to the planes of equal amplitude the amplitude decreases over a distance x in ratio 1 to $e^{-2\pi kx : \lambda}$. As $q_1 = 0$, we get according to (8) and (9):

$$\frac{2\pi k x}{\lambda} = \frac{2\pi \varrho x}{\lambda} (n_0\, sin\,\omega + k_0\, cos\,\omega)$$

from which again follows, when $cot\,\tau$ is substituted for $n_0 : k_0$:

$$k = k_0\, \varrho\, sin\,(\tau + \omega) : sin\,\tau$$

or on account of (3):

$$k = \sigma\, \varrho\, sin\,(\tau + \omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

5. The fundamental equations follow immediately from the values found for the index of refraction and the coefficient of absorption. The equations (17) and (18) lead immediately to:

$$n^2 - k^2 = \sigma^2\, \varrho^2\, cos\,2\,(\tau + \omega) + sin^2\, i \, .$$

According to (1) the second member of this equation is equal to $\sigma^2\, cos\,2\tau$ or according to (3) to $n_0^2 - k_0^2$. In this way the first fundamental equation is obtained.

Further follows from (15), (17) and (18):

$$n\,k\, cos\,\alpha = \frac{1}{2}\, \sigma^2\, \varrho^2\, sin\,2\,(\tau + \omega).$$

According to (2) the second member is equal to $\sigma^2\, sin\,2\tau$ and so according to (3) to $n_0\, k_0$, and thus the second equation has also been derived.

To conclude we may remark, that here the reversed course has been taken from that by which in the preceding paper the occurrence of the so-called complex index of refraction was derived from the two fundamental equations [1]).

Mathematics. — "*A tortuous surface of order six and of genus zero in space Sp_4 of four dimensions.*" By Prof. P. H. Schoute.

1. We begin by putting the following question:

"In space Sp_4 are given three planes $\alpha_1, \alpha_2, \alpha_3$ and in these are "assumed three projectively related pencils of rays. We demand the "locus of the common transversal of the triplets of rays corresponding "to each other."

Notation. We indicate the vertices of the rays of pencils by O_1, O_2, O_3, three corresponding rays and their transversal by l_1, l_2, l_3 and l, the points of intersection of l and l_1, l_2, l_3 by S_1, S_2, S_3 and the pencils of rays by $(l_1), (l_2), (l_3)$. Let further P_{23}, P_{13}, P_{12} indicate the points of intersection of the planes $\alpha_1, \alpha_2, \alpha_3$ two by two, and α the plane $P_{23}\, P_{13}\, P_{12}$ which has a line in common with each of the planes $\alpha_1, \alpha_2, \alpha_3$, namely with α_1 the line $P_{13}\, P_{12} = a_1$, with α_2 the line $P_{12}\, P_{23} = a_2$, with α_3 the line $P_{23}\, P_{13} = a_3$. We take for granted that not one of the three vertices O_1, O_2, O_3 coincides with one of the points P_{23}, P_{13}, P_{12}.

2. The answering of the given question offers no more difficulties, as soon as the locus of point S_1 in α_1 is known; so we shall first find this. Each ray l_1 of pencil (l_1) furnishing a single point S_1, it is a rational curve, whose degree surpasses the number of times a transversal l passes through O_1 with unity. Now two transversals l pass through O_1. For the pencil of planes $(O_1\, l_2)$ with $(O_1\, \alpha_2)$ as bearing space and $O_1\, O_2$ as axis marks on the line of intersection m of $(O_1\, \alpha_2)$ with α_3 a series of points (P) projectively related to the pencil of rays (l_3), from which ensues that there are two rays l_3 passing through their corresponding point P and that therefore there

[1]) See loc. cit. § 5.

are two lines $O_1 P$ cutting the corresponding line l_3, i.e. that two transversals l pass through O_1. So the locus of S_1 is a cubic curve s_1^3 having O_1 as node, and so we find for the loci of S_2 and S_3 in α_2 and α_3 in the same way rational curves s_2^3 and s_3^3 with O_2 and O_3 as nodes.

3. To determine the degree of the scroll of the lines l we first investigate what this scroll has in common with an arbitrary space through α_1. Each point Q lying outside α_1 which this space has in common with the scroll gives a line l having two points in common with that space, therefore lying entirely in that space. So that space can contain besides s_1^3 only a certain number of generatrices l of the scroll. As the line of intersection of α_2 with the assumed space through α_1 has three points in common with s_2^3 the number of generatrices to be found is three and the scroll, having a system of lines of order six in common with the assumed space, must be a tortuous surface O^6 of order six. So it is cut by an arbitrary space according to a twisted curve of order six; this section in general not degenerating is rational, its points corresponding one by one to the lines l and therefore to the rays of each of the pencils $(l_1), (l_2), (l_3)$. So the surface is of genus zero.

We call the locus just found — however not yet what was meant in the title — a surface, to show by this that the number of points is twofold infinite; by the predicate "tortuous" we express that it is not situated in a threedimensional space.

4. By considering the three projective series of points $(A_1), (A_2), (A_3)$ marked by the three projective pencils of rays $(l_1), (l_2), (l_3)$ on a_1, a_2, a_3 we easily prove that the plane α contains three generatrices of O^6. For it happens, we know, three times that three corresponding points A_1, A_2, A_3 of the projective series of points $(A_1), (A_2), (A_3)$ lie in a same right line, which then becomes a generatrix l of O^6; for, the conics enveloped by the lines $A_1 A_2$ and $A_1 A_3$ connecting each point A_1 with the corresponding points A_2 and A_3 have besides a_1 still three common tangents.

To the rule that the tangents in a point of O^6 drawn to O^6 are situated in a plane, the points of intersection of two non-successive generatrices l form an exception. In such a point, through which the surface passes twice, a tangential plane will belong to each of the two lines l; so it can be called a "biplanar node". From the above is evident that O^6 possesses six biplanar nodes, the three points O_1, O_2, O_3 and the three points of intersection of the genera-

trices lying in α; moreover we shall see directly that the number of those nodes is in general six, becoming infinite when it surpasses six, as takes place in the surface to be considered presently and which is indicated in the title.

5. We point out the fact, that the found surface O^6 is determined by the projective correspondence of the curves s_1^3 and s_2^3 in α_1 and α_2, and we now show that this correspondence, characterised by the particularity of the corresponding triplets lying on a_1 and a_2, is not the most general one can think of. To that end we take two rational curves s_1^3 and s_2^3 in two planes α_1 and α_2, which planes for convenience' sake we assume for the present to be lying in our space, and which curves with the nodes O_1 and O_2 we suppose to be brought into projective correspondence in the most general manner. Are there then — we ask — to be found on s_1^3 three collinear points to which on s_2^3 three likewise collinear points correspond? The answer runs affirmatively; what is more: each point of s_1^3 forms one time a part of such a triplet and the bearing lines form a pencil of rays. If namely the point A_2 of s_2^3 corresponds to the point A_1 taken arbitrarily on s_1^3, and if the central involution of the points B_1, C_1 of s_1^3 collinear with A_1 is represented by $(B_1 C_1)$, the non-central involution of the corresponding points B_2, C_2 of s_2^3 by $(B_2 C_2)$ and the central involution of the points B_2', C_2' of s_2^3 collinear with A_2 by $(B_2' C_2')$, then the two involutions $(B_2 C_2), (B_2' C_2')$ have a pair of points in common. If B_2°, C_2° is this pair and B_1°, C_1° on s_1^3 the pair corresponding to it, then $A_1, B_1^\circ, C_1^\circ$ and $A_2, B_2^\circ, C_2^\circ$ are two corresponding collinear triplets. If now Q_1 is the point of intersection of two such like lines l_1', l_1'' in α_1 and Q_2 the point of intersection of the corresponding lines l_2', l_2'' in α_2, then the triple involution $(A_1 B_1 C_1)$ marked by the lines through Q_1 in s_1^3 must correspond to the triple involution $(A_2 B_2 C_2)$ marked by the lines through Q_2 in s_2^3, with which we have proved what was asserted above.

With the aid of the preceding it is easy to show in how far the particularity of the corresponding triplets lying on a_1 and a_2 is a real one or an apparent one. With respect to the planes α_1 and α_2 placed in our space it is evidently an apparent one; for not one time but an infinite number of times it happens that three collinear points of s_1^3 correspond to three likewise collinear points of s_2^3. If the planes α_1 and α_2 are placed in Sp_4 in such a way that an arbitrary point P_1 of α_1 coincides with an arbitrary point P_2 of α_2, then however the three points in which s_1^3 is cut by the line $P_1 Q_1$ will correspond to three collinear points of s_2^3, but the line through Q_2

bearing the last three points will in general not pass through $P_2 = P_1$. So then there are no lines a_1 and a_2 to be drawn through the point of intersection P_{12} of the planes α_1 and α_2 cutting s_1^3 and s_2^3 in corresponding triplets.

6. We shall now consider the more general case of two projectively related curves s_1^3 and s_2^3 lying in such a way in α_1 and α_2 that through the point of intersection P_{12} no triplets of corresponding points bearing lines a_1 and a_2 are to be drawn. The argument leading to the order of the scroll, which is the locus of the line connecting the corresponding points of those curves, retains here its force. So we have but to determine the number of nodes. Of course O_1 and O_2 are nodes. If farthermore A_1A_2 and B_1B_2 are two generatrices cutting each other outside α_1 and α_2, then A_1B_1 and A_2B_2 pass through P_{12}, as they must cut each other. So we consider the central triple involution $(A_1\, B_1\, C_1)$ marked by the pencil of rays with P_{12} as vertex in s_1^3 and the non-central triple involution $(A_2\, B_2\, C_2)$ of the corresponding triplets of s_2^3; then the latter furnishes as envelope of the sides of the triangles $A_2\, B_2\, C_2$ a definite curve of involution which makes us acquainted by the number of its tangents through P_{12} with the number of nodes not lying in α_1 and α_2 of the new surface O^6. Now the class of the indicated curve of involution is four; for evidently four tangents pass through the node O_2 of s_2^3. If to the two points of s_2^3 coinciding in O_2 the points M_1, N_1 on s_1^3 correspond, and if $P_{12}\, M_1$ and $P_{12}\, N_1$ cut the curve s_1^3 still in the point M_1', M_1'' and N_1', N_1'', then the lines connecting P_{12} with the corresponding points M_2', M_2'', N_2', N_2'' are the only tangents of the curve of involution passing through P_{12}. So O^6 has here also six nodes.

7. It is now easy to see that the first surface O^6 of the three projective pencils of rays is found back, if the correspondence of the curves s_1^3 and s_2^3 is given in such a way that through the point of intersection P_{12} of α_1 and α_2 lines a_1 and a_2 pass bearing two triplets of corresponding points. The plane a_1a_2 is then again a plane α through three generatrices of O^6 and the line a_2 represents three of the four tangents to be drawn through P_{12} to the above found curve of involution, whilst the fourth tangent causes us to find a node not lying in α_1, α_2 or α. If we now cut the surface by the space determined by α and this node, the section will consist of the three generatrices in α and a curve of order three with a node, i. e. a rational plane cubic curve. The plane of that curve is then

the plane α_3, the node of that curve the point O_3 of the first generation.

8. We now ask what arises when the planes α_1 and α_2 are placed in such a way in Sp_4, that the points Q_1 and Q_2 coincide and therefore each line drawn through the point of coincidence P_{12} in α_1 is to be regarded as line a_1. We then find that to every line a_1 through P_{12} in α_1 a definite line a_2 through P_{12} in α_2 corresponds, so that there is an infinite number of planes α. The locus of these planes α is a quadratic conic space with P_{12} as vertex; for the pencils of rays of the lines a_1, a_2 through P_{12} corresponding to each other in α_1, α_2 are evidently projectively related. This quadratic conic space must contain, as it contains all generatrices of O^6, this tortuous scroll itself.

Now that the generatrices of this particular surface O^6, being the surface indicated in the title, group themselves into triplets lying in a plane, there must be a locus of nodes. This is of order four. If namely we project the surface O^6 by means of the just found quadratic conic space out of P_{12} on to an arbitrary space not containing P_{12}, the projection is a quadratic scroll having the projections of the planes α as a system of generatrices. Of this surface O^2 the projections of α_1 and α_2 form thus two lines of the other system; for each of those two planes has a line in common with each of those planes α and from this ensues that the sections of α_1 and α_2 with the space of projection must have a point in common with the sections of the planes α with that space of projection. So in that space of projection each plane through one of the two lines contains a line of the system corresponding to the planes α and therefore the projections of four nodes, namely one on the first line and three on the second. So the projection of the nodal curve out of P_{12} on to the assumed space of projection is a curve of order four lying on O^2, which has *one* point in common with each of the generatrices of one system, and *three* points with each of the generatrices of the other system. So the nodal curve itself is a tortuous curve of order four; it is rational as its projection is.

Considering the surface O^2 we see at the same time that the surface O^6 admits of an infinite number of planes cutting it according to a rational cubic curve, namely each plane through P_{12} and one of the lines of the system to which the projections of α_1 and α_2 belong.

So we find the following theorem:

"If we assume in two planes α_1 and α_2 two projectively related "rational cubic curves, if in these planes we determine the vertices "Q_1, Q_2 of the corresponding central triple involutions on those "curves and if now we place those planes in S_4 in such a way that "Q_1 and Q_2 coincide in the points of intersection P_{12} of the planes, "the locus of the line connecting the pairs of corresponding points of "the cubic curves forms a tortuous surface with the following properties:

a. "It is projected out of P_{12} by a quadratic conic space, on "which two systems of planes are lying;

b. "It is cut by each plane of one system according to a cubic "curve with a node, by each plane of the other system according "to three generatrices;

c. "The cubic curves in two planes of the first system have no "point in common, neither have the triplets of lines in two planes of "the second systems; each cubic curve, however, is cut by each "generatrix;

d. "The generatrices cause a mutual projective correspondence "among all cubic curves and the cubic curves among all the generatrices."

9. From the preceding ensues immediately that the tortuous scroll with a nodal curve k^4 can be represented on a plane. If in a plane σ we assume arbitrarily two pencils of rays with different vertices T_1, T_2, and if we allow three arbitrary rays a_1, b_1, c_1 of the former to correspond to three rational cubic curves $s^3_{(a)}$, $s^3_{(b)}$, $s^3_{(c)}$ of O^6, three arbitrary rays a_2, b_2, c_2 of the second to three generatrices $l_{(a)}$, $l_{(b)}$, $l_{(c)}$ of O^6, then to each rational cubic curve $s^3_{(p)}$ corresponds a definite ray p_1 of the first pencil, to each generatrix $l_{(q)}$ corresponds a definite ray q_2 of the second; so we can assign the point of intersection of $s^3_{(p)}$ and $l_{(q)}$ to the point of intersection of p_1 and q_2. The elements of exception of that representation are immediately found. If to the line connecting the vertices of the pencils of rays counted with the first pencil the curve s^3 corresponds and counted with the second pencil the generatrix l, and if S is the point of intersection of s^3 and l, then to point T_1 corresponds the line l, to point T_2 the curve s^3 and reversely to point S the line $T_1\,T_2$. To each point P of the nodal curve k^4 correspond two points P', P'' of σ collinear to T_1, because in the correspondence of $s^3_{(p)}$ to $l_{(p)}$ the node of $s^3_{(p)}$ represents two different points and two points of $l_{(p)}$ correspond to this point. As T_1 forms part of two representing pairs, the pairs belonging to the nodes of the generatrix l belonging to T_1, this point is node and the curve of order four. This is also evident when we consider the rays of the other pencil. On each ray q lie two points of the curve forming

part of pairs corresponding to the points of the nodal curve lying on $l_{(q)}$, whilst T_2 corresponding to the node of the curve s^3 is likewise node of the curve. So in σ to the nodal curve k^4 corresponds a curve $k'^4(T_1^2, T_2^2)$, having T_1 and T_2 as nodes and being of genus *unity*. And to the rational space sections k^6 of O^6 correspond in σ curves $k'^4(T_1, T_2^3)$ through T_1 with T_2 as triple point, which is found immediately when we remember that an arbitrary space has three points in common with each of the rational cubic curves of O^6 and one point with each generatrix of O^6. As is proper each of those rational curves $k'^4(T_1, T_2^3)$ has with the representation $k'^4(T_1^2, T_2^2)$ of the nodal curve k^4 besides T_1 and T_2 four pairs of points in common, corresponding to the four points of the nodal curve lying in the selected space, whilst two curves $k'^4(T_1, T_2^3)$ cut each other besides in T_1 and T_2 in six points corresponding to the six points of intersection of O^6 with the plane of section of the two spaces.

10. The locus of the bisecants of a tortuous curve of order four is a curved space of order three having the indicated curve as nodal curve. For the twofold infinite number of bisecants furnishes a triple infinite number of points and three of these lie on an arbitrary right line l, because the curve projects itself out of l on to a plane not intersecting l as a rational curve of order four and this plane curve possesses three double points. If we apply this to the nodal curve k^4 of O^6, taking into consideration that the generatrices of this scroll are all bisecants of k^4, we find:

"The tortuous scroll O^6 with the nodal curve k^4 is the complete section of a quadratic conic space with a curved space of order three, of which the first passes once, the second twice through k^4."

Whilst the cubic space is the locus of the bisecants of k^4, the quadratic conic space with P_{12} as vertex is the locus of the planes containing three points of k^4 and passing through P_{12}.

The tortuous surface O^6 with a nodal curve k^4 is determined by this curve and the point P_{12}. As P_{12} lies arbitrarily with respect to k^4 each tortuous curve k^4 in Sp_4 is nodal curve of a fourfold infinite number of surfaces O^6.

11. We observe that the case just considered of the correspondence of the curves s_1^3 and s_2^3, where a tortuous scroll with a double curve k^4 is formed, is not the most particular one that one can think of. If for instance — instead of starting from two rational curves s_1^3 and s_2^3 taken arbitrarily in α_1 and α_2 — we start by making the pointfields α_1 and α_2 to be in projective correspondence and then

continue to assume two rational curves s_1^3 and s_2^3 corresponding to each other in this manner, then of course to every three collinear points of s_1^3 correspond three likewise collinear points of s_2^3, and therefore we can take for the above determined pair of points Q_1, Q_2 *any* corresponding pair of points of α_1, α_2. In this special case a plane α through three generatrices will present itself already for arbitrary position in Sp_4 and the position, that an infinite number of those planes present themselves, will be able to be brought about in a twofold infinite number of different ways; in the last case however the three generatrices lying in a plane α pass through a point, as the series of points lying on the lines of intersection a_1, a_2 of this plane with α_1, α_2 are perspectively related, so that the locus of the nodes becomes a *conic* instead of a k^4. In both cases surfaces O^6 are formed differing from the above also in this respect that they admit not only of a single but of a twofold infinite number of spaces through three generatrices.

12. Also when we start from two projective rational curves s_1^3, s_2^3 in not projectively related fields a great number of special cases are left for consideration. So the point of intersection P_{12} of the planes α_1, α_2 can lie

a. on one of the curves s^3,

b. on both curves s^3,

c. on the two curves s^3 and correspond to itself,

d. it can be the node of one of the curves s^3,

e. it can be the node of one of the curves and lying on the other,

f. it can be the node of one of the curves and forming on the other part of the two points corresponding to this node,

g. it can be the node of both curves,

h. it can be the node of both curves and in such a way that one pair of points coinciding in this node has a point in common with the other,

i. it can be the node of both curves and in such a way that the pairs of points coinciding in this point correspond to each other.

Of course the number is still increased if we further permit the pointfields α_1, α_2 to be projectively related. We do not wish to investigate more closely all these special cases. Neither do we intend to investigate here the scrolls presenting themselves in both cases of projective or non-projective pointfields α_1, α_2 as the locus of the line P_1P_2 connecting corresponding points P_1, P_2 of *other* curves of the same genus and of the same order, which are projectively related. We only wish to observe that these scrolls will lie in the

case of the projectively related pointfields α_1, α_2 on the locus of the line P_1P_2 connecting corresponding points P_1, P_2 of the planes α_1, α_2, which is a quadratic or a cubic space according to the point of intersection P_{12} of α_1 and α_2 corresponding to itself or not.

13. We conclude with the deduction of the equations of the above found cubic and quadratic spaces which have in common the surface O^6 with the nodal curve k^4 and to this end we start from this curve. If the curve k^4 is given by the system of equations:

$$\varrho x_i = \lambda^i, \; (i = 0, 1, 2, 3, 4). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

— and in this way the simplex of coordinates can always be taken —, and if the point which is the vertex of the quadratic conic space with respect to that same simplex has the coordinates $(y_0, y_1, y_2, y_3, y_4)$, then the equations

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0 \;, \quad \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = 0 \quad . \quad . \quad . \quad (2)$$

represent those two spaces. We see namely immediately that the first determinant by insertion of the relations (1) shows three equal rows, i.e. that the cubic space represented by the first equation must have the points of the curve k^4 as nodes, and must thus contain each bisecant of k^4. Further it is equally clear that the second determinant by insertion of the relations (1) shows two equal rows and that, when substituting y_i for x_i, two pairs of equal rows appear, from which ensues that the quadratic space represented by the second equation passes through k^4 and has a node in y.

A more direct deduction of the equation of the locus of the bisecants of the curve k^4 was communicated formerly (*Proceedings* of the February meeting of 1899 vol. I, page 313). It is founded on the wellknown lemma, according to which the product of two matrices $M_1^{r,k}$ and $M_2^{r,k}$ with r rows and k columns, taken according to the rows, vanishes identically for $r > k$. This same lemma leads to the deduction of the equation of the locus of the planes containing three points of k^4, and passing through $(y_0, y_1, y_2, y_3, y_4)$. An arbitrary point P of the plane $P_1 P_2 P_3$ through the points P_1, P_2, P_3 of k^4 corresponding to the parametervalues λ_1, λ_2, λ_3 is represented by

$$\varrho \, x_i = p_1 \lambda_1^i + p_2 \lambda_2^i + p_3 \lambda_3^i, \quad (i = 0, 1, 2, 3, 4) \quad . \quad . \quad . \quad (3)$$

If the plane $P_1 P_2 P_3$ passes moreover through the given point $y_0, y_1, y_2, y_3, y_4)$, also the relations

$$\sigma y_i = q_1 \lambda_1^i + q_2 \lambda_2^i + q_3 \lambda_3^i, \quad (i = 0, 1, 2, 3, 4) \quad . \quad . \quad . \quad (4)$$

hold, and now the equation sought for is found by eliminating the nine quantities $\lambda_1, \lambda_2, \lambda_3, p_1, p_2, p_3, q_1, q_2, q_3$ out of the ten equations (3) and (4). This takes place by inserting the values given by (3) and (4) in the left hand member of the second equation (2). For by this we find

$$\varrho^2 \sigma^2 \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{vmatrix} \cdot \begin{vmatrix} p_1 & p_2 & p_3 \\ p_1\lambda_1 & p_2\lambda_2 & p_3\lambda_3 \\ q_1 & q_2 & q_3 \\ q_1\lambda_1 & q_2\lambda_2 & q_3\lambda_3 \end{vmatrix} = 0.$$

We considered in the above cited communication equations forming the extension of the first of the equations (2) to the curve k^{2n} of the space Sp_{2n}. In connection with this we shall notice that the second of the equations (2) admits of corresponding extensions, in which those of the first are included. However, these will be developed elsewhere.

Mathematics. — "*The* PLÜCKER *equivalents of a cyclic point of a twisted curve.*" By W. A. VERSLUYS. (Communicated by Prof. P. H. SCHOUTE.

If a twisted curve C admits of a higher singularity (cyclic point) of order n, of rank r and of class m, it is to be represented according to HALPHEN [1]) in the vicinity of this singular point M by the following developments in series:

$$x = t^n,$$
$$y = t^{n+r}\,[t],$$
$$z = t^{n+r+m}\,[t],$$

where $[t]$ represents an arbitrary power series of t, starting with a constant term.

If n, r and m satisfy the conditions that

$$\begin{array}{ll} 1^\circ & n \text{ and } r, \\ 2^\circ & r \text{ and } m, \\ 3^\circ & n \text{ and } r+m, \\ 4^\circ & n+r \text{ and } m \end{array} \qquad (A)$$

are mutually prime, then this higher singularity $M\ (n, r, m)$ for

[1]) Bull. d. l. Soc. Mat. d. France t. VI p. 10.

the formulae of CAYLEY-PLÜCKER and for the genus is equivalent to the following numbers of ordinary singularities:

$$\begin{array}{l} n-1 \text{ cusps } \beta, \\ \dfrac{(n-1)(n+r-3)}{2} \text{ nodes } H, \\ m-1 \text{ stationary planes } \alpha, \\ \dfrac{(m-1)(m+r-3)}{2} \text{ double planes } G, \\ r-1 \text{ stationary tangents } \theta, \\ \dfrac{(r-1)(r+m-3)}{2} \text{ double generatrices } \omega_1, \\ \dfrac{(r-1)(r+n-3)}{2} \text{ double tangents } \omega_2. \end{array} \qquad (B)$$

For a curve with only ordinary singularities we always have $\omega_1 = \omega_2$.

If the curve admits of higher singularities, then the tangents in these singular points will not have to count for as many double tangents to the curve as they must count for double generatrices of the developable belonging to the curve. The number ω will then be different for the formulae of CAYLEY-PLÜCKER, relating to a section and for those formulae relating to a projection, i. o. w. the singularity ω of a twisted curve appearing in a term $(x + \omega)$ is not always the same as the one appearing in the term $(y + \omega)$.

So the formula

$$y - x = v - \mu \ ^{1})$$

is no longer correct as soon as the curve has higher singularities for which order and class are unequal.

The above as well as the following results do not hold for a common cusp $\beta\,(2, 1, 1)$ and for a common stationary plane $\alpha\,(1, 1, 2)$, the conditions (A) not being satisfied for these cyclic points.

Through the singular point $M\ (n, r, m)$ pass

$$\frac{n\,(n+2r+m-4)}{2}$$

branches of the nodal curve of the developable O belonging to the curve C.

All these branches touch the curve C in M and have in M with the common tangent

$$\frac{(n+r)\,(n+2r+m-4)}{2}$$

coinciding points in common.

[1]) SALMON. 3 Dim. § 327.