Mathematics. — "*The quotient of two successive Bessel Functions.*"
By Prof. W. KAPTEYN.

If $I^{\nu+1}(z)$ and $I^{\nu}(z)$ represent two successive Bessel Functions of the first kind, the quotient may be expanded as follows:

$$\frac{I^{\nu+1}(z)}{I^{\nu}(z)} = f_1 z + f_2 z^3 + f_3 z^5 + \ldots$$

Of course this equation holds for all values of z within a circle whose radius is equal to the modulus of the first root of the equation $I^{\nu}(z) = 0$, zero excepted. EULER and JACOBI have determined the first coefficients of this expansion; we wish to determine the general coefficient.

Starting from the known development

$$\frac{I^{\nu+1}(z)}{I^{\nu}(z)} = \cfrac{z}{2(\nu+1) - \cfrac{z^2}{2(\nu+2) - \cfrac{z^2}{2(\nu+3) - \text{etc.}}}}$$

and putting

$$z^2 = -x \qquad 2(\nu + p) = a_p$$

the question reduces to the determination of the general coefficient in the following equation:

$$\cfrac{x}{a_1 + \cfrac{x}{a_2 + \text{etc.}}} = f_1 x - f_2 x^2 + f_3 x^3 - \text{etc.}$$

Let $\frac{P_n}{Q_n}$ stand for the approximating fractions of the continued fraction in the first member, and let

$$\begin{aligned}
Q_{2n+1} &= \nu_0 + \nu_1 x + \nu_2 x^2 + \ldots + \nu_n x^n \\
Q_{2n} &= \mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_n x^n \\
Q_{2n-1} &= \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_{n-1} x^{n-1} \\
Q_{2n-2} &= \varkappa_0 + \varkappa_1 x + \varkappa_2 x^2 + \ldots + \varkappa_{n-1} x^{n-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
Q_{n+2} &= \zeta_0 + \zeta_1 x + \ldots + \zeta_r x^r \\
Q_{n+1} &= \varepsilon_0 + \varepsilon_1 x + \ldots + \varepsilon_s x^s
\end{aligned}$$

where

$$r = \frac{n}{2} + 1 \qquad s = \frac{n}{2}$$

when n even, and

$$r = \frac{n+1}{2} \qquad s = \frac{n+1}{2}$$

when n is odd, then we find

$$a_1^{n+1}\, a_2^{n}\, a_3^{n-1} \dots a_n^2\, a_{n+1} f_{n+1} = (-1)^h \begin{vmatrix} \lambda_1 & \varkappa_1 & \iota_1 & \theta_1 \dots \zeta_1 & \varepsilon_1 \\ \lambda_2 & \varkappa_2 & \iota_2 & \theta_2 \dots \zeta_2 & \varepsilon_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2} & \varkappa_{n-2} & \iota_{n-2} & \theta_{n-2} \dots 0 & 0 \\ \lambda_{n-1} & \varkappa_{n-1} & 0 & 0 \dots 0 & 0 \end{vmatrix}$$

In this equation h stands for $\frac{n}{2} - 1$ if n is even and for $\frac{n-1}{2}$ when n is odd. If now we replace a_p by $2(\nu + p) = 2b_p$ we obtain the following results. Firstly

$$2^{2n+1}\, b_1^{n+1}\, b_2^{n-1} \dots b_{\frac{n}{2}+1}\, b_{\frac{n}{2}+2} \dots b_n^2\, b_{n+1} f_{n+1} =$$

$$= (-1)^{\frac{n}{2}-1} \begin{vmatrix} \lambda_1' & \varkappa_1' & \iota_1' & \dots & \varepsilon_1' \\ \lambda_2' & \varkappa_2' & \iota_2' & \dots & \varepsilon_2' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_{\frac{n}{2}}' & \varkappa_{\frac{n}{2}}' & \iota_{\frac{n}{2}}' & \dots & \varepsilon_{\frac{n}{2}}' \\ \lambda_{\frac{n}{2}+1} & \varkappa_{\frac{n}{2}+1} & \iota_{\frac{n}{2}+1} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2} & \varkappa_{n-2} & \iota_{n-2} & \dots & 0 \\ \lambda_{n-1} & \varkappa_{n-1} & 0 & \dots & 0 \end{vmatrix} \quad \dots \quad (I)$$

if n is an even number, and secondly

$$2^{2n+1}\, b_1^{n+1}\, b_2^{n+1} \dots b^2_{\frac{n+1}{2}}\, b_{\frac{n+3}{2}} \dots b_n^2\, b_{n+1} f_{n+1} =$$

$$= (-1)^{\frac{n-1}{2}} \begin{vmatrix} \lambda_1' & \varkappa_1' & \iota_1' & \dots & \varepsilon_1' \\ \lambda_2' & \varkappa_2' & \iota_2' & \dots & \varepsilon_2' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_{\frac{n-1}{2}}' & \varkappa_{\frac{n-1}{2}}' & \iota_{\frac{n-1}{2}}' & \dots & \varepsilon_{\frac{n-1}{2}}' \\ \lambda_{\frac{n+1}{2}} & \varkappa_{\frac{n+1}{2}} & \iota_{\frac{n+1}{2}} & \dots & \varepsilon_{\frac{n+1}{2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2} & \varkappa_{n-2} & \iota_{n-2} & \dots & 0 \\ \lambda_{n-1} & \varkappa_{n-1} & 0 & \dots & 0 \end{vmatrix} \quad \dots \quad (II)$$

if n is an odd number, where

$$\lambda'_p = \frac{(2n-p-1)(2n-p-2)\dots(2n-2p)}{p!} b_{n-p+2} \dots b_{2n-p-1}$$

$$\varkappa'_p = \frac{(2n-p-2)(2n-p-3)\dots(2n-2p-1)}{p!} b_{n-p+2} \dots b_{2n-p-2}$$

$$\iota'_p = \frac{(2n-p-3)(2n-p-4)\dots(2n-2p-2)}{p!} b_{n-p+2} \dots b_{2n-p-3}$$

. . . .

$$\varepsilon'_p = \frac{(n-p+1)(n-p)\dots(n-2p+2)}{p!}$$

en

$$\lambda_p = \frac{(2n-p-1)(2n-p-2)\dots(2n-2p)}{p!} b_{p+1} \dots b_{2n-p-1}$$

$$\varkappa_p = \frac{(2n-p-2)(2n-p-3)\dots(2n-2p-1)}{p!} b_{p+1} \dots b_{2n-p-2}$$

$$\iota_p = \frac{(2n-p-3)(2n-p-4)\dots(2n-2p-2)}{p!} b_{p+1} \dots b_{2n-p-3}$$

. . . .

$$\varepsilon_p = \frac{(n-p+1)(n-p)\dots(n-2p+2)}{p!} b_{p+1} \dots b_{n-p+1}.$$

It is of importance to remark that

$$\lambda_{n-1} = nb_n, \quad \varkappa_{n-1} = 1, \quad \iota_{n-2} = (n-1)\, b_{n-1}, \quad \theta_{n-2} = 1 \text{ etc.}$$

and that the determinants in the second members of the equations (I) and (II) after the substitution $b_p = \nu + p$, are respectively polynomia of degrees $\frac{n(n-2)}{4}$ and $\frac{(n-1)^2}{4}$ in ν.

Meteorology. – "*On frequency curves of barometric heights.*" By Dr. J. P. van der Stok.

1. The records of barometric heights, corrected for temperature, observed at Helder three times a day during the years August 1843 to July 1904, have been chosen as an appropriate material for this inquiry into the nature of barometric frequency curves. The number of observations for each month amounts to:

January	5673	July	5673
February	5169	August	5766
March	5646	September	5560
April	5490	October	5766
May	5673	November	5580
June	5490	December	5766
		Total	67252