to the capitulum of the female or of the hermaphrodite is one at each side only, in some species it is two or three and the largest number I have observed was five. How can we explain that there is a species with such a large number as the case mentioned? I have tried in vain to find an explanation. We do not know much of the habits of these animals. It is hardly admissible that the great number of males should be connected with the depth at which they live, for (1) the same species which is found in the Malay Archipelago at a depth of 200—400 m. lives in the Japan sea in shallow water, and (2) we know species living in coastal waters and others found in depths of over 1800 m., all of which have two males only. A connection exists no doubt between the place where the little males are found attached and their great number – but I am at a loss to understand what the relation may be. The eggs of these Cirripedes are fecundated at the moment they are excluded and form two leaves (the so-called ovigerous lamellae) which remain in the sack or mantle-cavity of the female until the eggs hatch out. If the males are attached at the margin of the mantle-cavity, the chance that the eggs will be impregnated is of course larger than in the case when they are attached at a greater distance, as in *Sc. Stearnsi.* So it is easily understood that in the latter case a greater number of males would be required — but why did they choose for attachment a place which is less favourable for impregnation? Because they were so numerous and did not find space enough at the ordinary place?

**Mathematics.** — "*A group of complexes of rays whose singular surfaces consist of a scroll and a number of planes*". By Prof. Jan de Vries.

1. The generatrices of a rational scroll can be arranged in the groups of an involution $I_p$; to this end we have but to arrange their traces on an arbitrary plane in the groups of an $I_p$. If we consider each pair of lines $l, l'$ of $I_p$ as directrices of a linear congruence, it immediately occurs to us to examine the complex of rays $\Gamma$ which is the compound of the $\infty^1$ congruences determined by it.

Let the scroll $\varrho^n$ be of order $n$ and let it have an $(n-1)$-fold directrix $d$. The generatrices $l$ form a fundamental involution $I_{n-1}$, each group of which consists of the $(n-1)$ right lines, coinciding in a point of $d$. This $I_{n-1}$ has evidently $(n-2)(p-1)$ pairs in common with the given $I_p$; so on $d$ lie as many points of intersection $H$ of pairs $l, l'$ of the involution $I_p$. Each ray through a point $H$ belongs to the complex $\Gamma$, likewise each ray in the connecting plane $h$ of the right lines $l, l'$; i. o. w. the complex has $(n-2)(p-1)$ *principal points* $H$ and $(n-2)(p-1)$ *principal planes* $h$.

2. On an arbitrary plane $\alpha$ a rational curve $c^n$ with $(n-1)$-fold point $D$ is determined by $\varrho^n$. The rays of the complex lying in $\alpha$ envelop a curve $(\alpha)$ of class $(n-1)(p-1)$, the curve of involution (director curve) of $I'_p$ in which the points of $c^n$ are arranged by the given $I_p$. [1]

*So the complex is of order* $(n-1)(p-1)$.

The line of intersection of $\alpha$ with a principal plane $h$ being a ray of $\Gamma$, the curve of the complex $(\alpha)$ touches all principal planes.

If $\alpha$ is made to pass through a right line $l$ of $\varrho^n$, then $(\alpha)$ splits up into the pencils having for vertices the traces $L'$ of the $(p-1)$ rays conjugate to $l$ and into a curve of order $(n-2)(p-1)$, the curve of the involution of $I'_p$ on the curve $c^{n-1}$ which $\alpha$ has in common with $\varrho^n$ besides. So a tangent plane of $\varrho^n$ is a singular plane of $\Gamma$.

*The singular surface consists of a scroll and the principal planes.*

When a tangent plane contains one of the principal points it passes in general through the directrix $d$, therefore through all principal points. Then $(\alpha)$ splits up into $(n-2)(p-1)$ pencils $(H)$ and $(p-1)$ pencils $(L')$.

Of the $n-1$ generatrices $l$ through a point $H$, two, $l_0$ and $l'_0$, form a pair of $I_p$. If we bring $\alpha$ through one of the remaining right lines $l_k$ ($k=1$ to $n-3$), then $(\alpha)$ consists of $(p-1)$ pencils with vertices $L'_k$, the pencil $(H)$ and a curve of class $(n-2)(p-1)-1$.

In an arbitrary plane through $H$ the curve of the complex $(\alpha)$ consists of the pencil $(H)$ and a curve of order $(n-1)(p-1)-1$.

3. The rays of the complex through an arbitrary point $A$ envelop a cone $(A)$ of order $(n-1)(p-1)$ passing through the principal points.

If $A$ lies on $\varrho^n$ cone $(A)$ consists of $(p-1)$ concentric pencils and a cone of order $(n-2)(p-1)$.

If we assume $A$ in a principal plane $h$ then only *one* pencil separates itself from the cone of the complex.

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[1]) $I'_p$ has $(n-1)(p-1)$ pairs in common with the involution $I_n$ which an arbitrary pencil determines on $c^n$.

If $A$ is taken on the line of intersection of two planes $h$, two pencils are separated from the cone. Three pencils are obtained when $A$ is point of intersection of three principal planes.

If we take $A$ on the curve $c^{n-2}$ which a plane $h$ has in common with $\varrho^n$ then $(A)$ consists of $p$ concentric pencils and a cone of order $(n-2)(p-1)-1$.

If $A$ is a point of intersection of $\varrho^n$ with two principal planes the number of pencils evidently becomes $(p+1)$.

4. The curve of the complex $(a)$ is of order $(p-1)(2n+p-6)$ [1]. It possesses $\frac{1}{2}(p-1)(p-2)(n-2)$ threefold tangents [2]) which are transversals of as many triplets of right lines belonging to a group of $I_p$. The cone of the complex $(A)$ possessing evidently as many threefold edges, the scrolls each having three conjugate right lines $l$ as directrices form together a *congruence* $\gamma$ of *order* and *class* $\frac{1}{2}(p-1)(p-2)(n-2)$.

Each principal point $H$ is for this congruence *a singular point* of order $(p-2)$; the *singular cone* is broken up into $(p-2)$ pencils.

Each principal plane $h$ is a *singular plane* of order $(p-2)$ and contains $(p-2)$ pencils of rays of congruence.

5. The right lines resting on four lines $l$ belonging to a group of $I_p$ form a scroll enclosed in $\Gamma$, of which the order is going to be determined.

Each transversal $t$ of three conjugate right lines $l_1, l_2, l_3$ and the arbitrary right line $a$ intersects still $(n-3)$ generatrices $m$ of $\varrho^n$. To each of these right lines $m$ can be made to correspond the $(p-3)$ right lines $l'$ forming with $l_1, l_2, l_3$ a group of $I_p$.

To each ray $l'$ belong $(p-1)_3$ triplets $l_1, l_2, l_3$, so $2(p-1)_3$ transversals $t$ and therefore $2(n-3)(p-1)_3$ rays $m$.

The congruence $(1, 1)$ of the right lines resting on $m$ and $a$ has with the congruence $\gamma$ in common $(n-2)(p-1)(p-2)$ rays $t$, so that to $m$ are conjugate $(n-2)(p-1)(p-2)(p-3)$ right lines $l'$.

Now each transversal of four lines $l$ belonging to a group of $I_p$ evidently gives four coincidences of the correspondence $(l', m)$.

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[1]) The characteristic numbers of the curve of involution of an $I_p$ on a rational $c^n$ are found in the dissertation of JOH. A. VRIESWIJK JR. (Involuties op rationale krommen, Utrecht 1905, page 38).

[2]) See also my paper "Complexes of rays in relation to a rational skew curve" (These Proceedings, VI, page 12).

*Consequently the scroll of the transversals of quadruplets of the involution is of order* $\frac{1}{12}(p-1)(p-2)(p-3)(4n-9)$.

Each principal point and each principal plane of $\Gamma$ bears $\frac{1}{2}(p-2)(p-3)$ right lines of this scroll.

6. If $\varrho^n$ possesses also a single directrix $e$ all principal planes of $\Gamma$ pass through $e$ and the complex is in itself dual.

If $\varrho^n$ has a nodal curve $\sigma$ of order $\frac{1}{2}(n-2)(n-1)$ each generatrix $l$ rests in $(n-2)$ points on $\sigma$, and is thus cut by $(n-2)$ right lines $l'$. By this the generatrices are arranged in a symmetric correspondence of order $(n-2)$, having with $I_p$ given on $\varrho^n$ in common $(n-2)(p-1)$ points $H$. So the complex has again $(n-2)(p-1)$ principal points and as many principal planes.

In like manner the order of $\Gamma$ remains the same. But now the curve of the complex can break up on account of its plane containing two or three principal points by which two or three pencils are separated. Besides $\alpha$ can contain still a right line $l$. So here the degenerations of $(\alpha)$ are dually opposed to those of the cone $(A)$.

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