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Physics. — "*On the propagation of light in a biaxial crystal around a centre of vibration.*" By H. B. A. BOCKWINKEL. (Communicated by Prof. H. A. LORENTZ).

(Communicated in the Meeting of January 1906).

In the electromagnetic theory of light, it is of interest to determine the electromagnetic field in a crystal due to an action, taking place in a certain centre O. In order to fix the ideas, we shall assume, that in an element of space τ at the point O there are certain periodic electromotive forces (*E. M. F.*). There will then be a radiation of energy from O in every direction, the amount of which will depend on this direction with respect to that of the *E. M. F.* and to those of the axes of electric symmetry. Our object is to investigate this dependence, at least for points at a great distance from O. We might for this purpose use the results of GRÜNWALD[1]); this physicist however takes the equations in the form they assume for a rigid elastic body and does not operate with an *E. M. F.* as mentioned above; we shall therefore treat the problem independently. Our method will consist in reducing the question to one of plane waves, by using a formula, proved by Prof. LORENTZ. In this formula a continuous function of the coordinates is represented by an integral over the solid angles of all cones having their vertices in O and filling the whole space. If the *E. M. F.* is $\mathfrak{E}^e$ then

$$\mathfrak{E}^e = -\int \frac{1}{8\pi^2} \frac{\partial^2 \mathfrak{W}}{\partial n^2} d\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where dn is the element of a line of arbitrary direction within the cone $d\omega$ and $\mathfrak{W}$ a vector given by

$$\mathfrak{W} = \int \mathfrak{E}^e \, d\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the integral being taken over the plane, passing through the point considered, perpendicularly to n. Hence, $\mathfrak{W}$ depends on the coordinates, but in such a way as to be constant in every plane perpendicular to n. By (1) the original *E. M. F.* has now been decomposed into a great number of infinitely small vectors, the effect of which can easily be calculated, each of them being constant in planes of a certain direction. Thus we determine the field, produced by each of the elements of the integral (1) and then compose all the fields obtained in this way into one resulting field, which,

[1]) J. GRÜNWALD. Über die Ausbreitung der Wellenbewegungen in optisch zweiachsigen elastischen Medien, BOLTZMANN Festschrift (1904), p. 518.

according to the principle of superposition, will really be the one produced by the whole E. M. F. Each of the separate very small fields will consist in a propagation of plane waves having the same direction as the planes in which the corresponding element of the E. M. F. is constant. The problem will therefore indeed be reduced to one of plane waves.

§ 2. In order to find the small field, corresponding to a cone of definite direction, we shall take a system of coordinates OX', OY', OZ', the axis OZ' coinciding with the axis of the chosen cone and OX', OY' respectively with the two directions of the dielectric displacement, belonging to plane waves, normal to OZ'. The wave that has its dielectric displacement along OX' will be called "the first wave"; the other "the second wave".

Again we take a system of coordinates OX, OY, OZ, the axes of which coincide with the axes of electric symmetry. Denoting the components of the electric force along the first axes by $\mathfrak{E}_{x'}$, $\mathfrak{E}_{y'}$, $\mathfrak{E}_{z'}$ and supposing all quantities to contain the factor $e^{i\frac{2\pi}{T}t}$ we have to satisfy the following equations

$$\left.\begin{aligned}
\Delta\mathfrak{E}_{x'}-\frac{\partial}{\partial x'}(div.\mathfrak{E})&=-\frac{4\pi^2}{T^2c^2}\left[\varepsilon_{11}(\mathfrak{E}_{x'}+\mathfrak{E}^e_{x'})+\varepsilon_{12}(\mathfrak{E}_{y'}+\mathfrak{E}^e_{y'})+\varepsilon_{13}(\mathfrak{E}_{z'}+\mathfrak{E}^e_{z'})\right]\\
\Delta\mathfrak{E}_{y'}-\frac{\partial}{\partial y'}(div.\mathfrak{E})&=-\frac{4\pi^2}{T^2c^2}\left[\varepsilon_{12}(\mathfrak{E}_{x'}+\mathfrak{E}^e_{x'})+\varepsilon_{22}(\mathfrak{E}_{y'}+\mathfrak{E}^e_{y'})+\varepsilon_{23}(\mathfrak{E}_{z'}+\mathfrak{E}^e_{z'})\right]\\
\Delta\mathfrak{E}_{z'}-\frac{\partial}{\partial z'}(div.\mathfrak{E})&=-\frac{4\pi^2}{T^2c^2}\left[\varepsilon_{13}(\mathfrak{E}_{x'}+\mathfrak{E}^e_{x'})+\varepsilon_{23}(\mathfrak{E}_{y'}+\mathfrak{E}^e_{y'})+\varepsilon_{33}(\mathfrak{E}_{z'}+\mathfrak{E}^e_{z'})\right]
\end{aligned}\right\}\quad(3)$$

It will not give rise to any misunderstanding that we have denoted here by $\mathfrak{E}^e$ the expression $-\frac{d\omega}{8\pi^2}\frac{\partial^2\mathfrak{B}}{\partial z'^2}$.

The quantities ε, occurring in these formulae, have particular properties, because they relate to *special* directions. These properties will show themselves in the following development. Since, according to the preceding considerations, $\mathfrak{E}^e$ depends only upon z', we shall find for $\mathfrak{E}$ a solution, likewise containing only z'. By this hypothesis the equations (3) become

$$\left.\begin{aligned}
\frac{\partial^2\mathfrak{E}_{x'}}{\partial z'^2}&=-\frac{4\pi^2}{c^2T^2}\left[\varepsilon_{11}(\mathfrak{E}_{x'}+\mathfrak{E}^e_{x'})+\varepsilon_{12}(\mathfrak{E}_{y'}+\mathfrak{E}^e_{y'})+\varepsilon_{13}(\mathfrak{E}_{z'}+\mathfrak{E}^e_{z'})\right]\\
\frac{\partial^2\mathfrak{E}_{y'}}{\partial z'^2}&=-\frac{4\pi^2}{c^2T^2}\left[\varepsilon_{12}(\mathfrak{E}_{x'}+\mathfrak{E}^e_{x'})+\varepsilon_{22}(\mathfrak{E}_{y'}+\mathfrak{E}^e_{y'})+\varepsilon_{23}(\mathfrak{E}_{z'}+\mathfrak{E}^e_{z'})\right]\\
0&=\varepsilon_{13}(\mathfrak{E}_{x'}+\mathfrak{E}^e_{x'})+\varepsilon_{23}(\mathfrak{E}_{y'}+\mathfrak{E}^e_{y'})+\varepsilon_{33}(\mathfrak{E}_{z'}+\mathfrak{E}^e_{z'})
\end{aligned}\right\}\quad(4)$$

§ 3. The last equation of (4) shows, that there is no dielectric displacement in the z'-direction. Further it is evident from these equations, that $\mathfrak{E}^e_{z'}$ has no share in the disturbance of the state of the aether at a distant point. Indeed, $\mathfrak{E}^e_{x'}$ and $\mathfrak{E}^e_{y'}$ being zero, the equations are satisfied by the solution

$$\mathfrak{E}_{x'} = 0, \quad \mathfrak{E}_{y'} = 0, \quad \mathfrak{E}_{z'} = -\mathfrak{E}^e_{z'}.$$

At the distant point $\mathfrak{E}^e_{z'}$ is zero, therefore $\mathfrak{E}_{z'}$ is so likewise. Electromotive forces acting within a layer bounded by two parallel planes and directed perpendicularly to these planes, do not therefore produce any disturbance of equilibrium at a distant point.

We eliminate $\mathfrak{E}_{z'}$ between the first and the third and between the second and the third equation.

This gives

$$\frac{\partial^2 \mathfrak{E}_{x'}}{\partial z'^2} = -\frac{4\pi^2}{c^2 T^2}\left[\left(\varepsilon_{11} - \frac{\varepsilon_{13}^2}{\varepsilon_{33}}\right)(\mathfrak{E}_{x'} + \mathfrak{E}^e_{x'}) + \left(\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{23}}{\varepsilon_{33}}\right)(\mathfrak{E}_{y'} + \mathfrak{E}^e_{y'})\right]$$

$$\frac{\partial^2 \mathfrak{E}_{y'}}{\partial z'^2} = -\frac{4\pi^2}{c^2 T^2}\left[\left(\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{23}}{\varepsilon_{33}}\right)(\mathfrak{E}_{x'} + \mathfrak{E}^e_{x'}) + \left(\varepsilon_{22} - \frac{\varepsilon_{23}^2}{\varepsilon_{33}}\right)(\mathfrak{E}_{y'} + \mathfrak{E}^e_{y'})\right].$$

According to what has already been said, these equations, if no E. M. F. are acting, must have one solution in which $\mathfrak{E}_{x'}$ is zero, and another in which $\mathfrak{E}_{y'}$ vanishes. This would follow from the equations themselves, if we knew the above mentioned properties of the quantities ε, occurring in them. Conversely, we shall be able to deduce these properties from the knowledge that the two solutions must satisfy the equations. Indeed these solutions can only hold if

$$\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{23}}{\varepsilon_{33}} = 0$$

and

$$V_{x'}^2 = \frac{c^2}{\varepsilon_{11} - \frac{\varepsilon_{13}^2}{\varepsilon_{33}}}, \quad V_{y'}^2 = \frac{c^2}{\varepsilon_{22} - \frac{\varepsilon_{23}^2}{\varepsilon_{33}}}$$

where $V_{x'}$ and $V_{y'}$ are the velocities of the plane waves in the two cases. By this the equations take the form

$$\frac{\partial^2 \mathfrak{E}_{x'}}{\partial z'^2} = -\frac{4\pi^2}{T^2 V_{x'}^2}(\mathfrak{E}_{x'} + \mathfrak{E}^e_{x'}), \quad \frac{\partial^2 \mathfrak{E}_{y'}}{\partial z'^2} = -\frac{4\pi^2}{T^2 V_{y'}^2}(\mathfrak{E}_{y'} + \mathfrak{E}^e_{y'}) \quad . \quad (5)$$

whereas the third equation of (4) gives $\mathfrak{E}_{z'}$ when $\mathfrak{E}_{x'}$ and $\mathfrak{E}_{y'}$ are known. We see from (5) that $\mathfrak{E}_{x'}$ depends only on $\mathfrak{E}^e_{x'}$, and $\mathfrak{E}_{y'}$ only on $\mathfrak{E}^e_{y'}$, further that both equations have the same form. We can therefore confine ourselves to considering only the first, in doing so we shall write V instead of $V_{x'}$. We shall have to remember however that after having found the result that is due to the X'-components of the E. M. F. we have still to add to this a second amount given by the Y'-component; this amount can be written down at once by analogy with the first.

§ 4. The general solution of the equation

$$\frac{\partial^2 \mathfrak{E}_{x'}}{\partial z'^2} = - \frac{4\pi^2}{T^2 V^2} (\mathfrak{E}_{x'} + \mathfrak{E}^e_{x'})$$

is given by

$$\mathfrak{E}_{x'} = \frac{i\pi}{TV} e^{i\frac{2\pi z'}{TV}} \int_{g_2}^{z'} \mathfrak{E}^e_{x'} e^{-i\frac{2\pi z'}{TV}} dz' - \frac{i\pi}{TV} e^{-i\frac{2\pi z'}{TV}} \int_{g_1}^{z'} \mathfrak{E}^e_{x'} e^{i\frac{2\pi z'}{TV}} dz' \quad . \quad (6)$$

The lower limit of these integrals is arbitrary, so that, as could be expected, two arbitrary constants occur in the solution. It is easily understood, that in the final result there will likewise be a certain indefiniteness. Indeed both a propagation towards O and one from O will be contained in it. It is sufficient for our present purpose to consider only the first solution and in order to leave aside the second we have to give completely definite values to the constants, as will appear in the following manner. We consider the two planes perpendicular to OZ', tangent to the boundary surface of the space τ; let these planes be determined by the equations

$$z' = - h_1 \quad \text{and} \quad z' = h_2 .$$

Then, since $\mathfrak{E}^e$ stands for

$$- \frac{1}{8\pi^2} \frac{\partial^2 \mathfrak{W}}{\partial z'^2} d\omega ,$$

it will differ from zero between the planes and will be zero in the space outside them. The first integral of (6) must vanish for

$$z' > h_2$$

and the second for

$$z' < - h_1 .$$

This is only possible, if

$$g_1 \leqq - h_1 \quad \text{and}$$

$$g_2 \geqq h_2 .$$

For the rest g_1 and g_2 may have any value satisfying these unequalities; it is evident that the result of the integrations will always be the same, if we take into account what has been said about the values of $\mathfrak{E}^e$. We shall therefore put $g_1 = -h_1$ and $g_2 = h_2$, so that

$$\mathfrak{E}_{x'} = \frac{i\,d\omega}{8\pi TV} e^{-i\frac{2\pi z'}{TV}} \int_{-h_1}^{z'} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{i\frac{2\pi z'}{TV}} dz' - \frac{i\,d\omega}{8\pi TV} e^{i\frac{2\pi z'}{TV}} \int_{h_2}^{z'} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{-i\frac{2\pi z'}{TV}} dz' \quad (7)$$

§ 5. In effecting these integrations we have to distinguish whether or no the point P, for which we intend to determine the state of radiation, lies between the two just mentioned tangent planes. First taking the latter case, the second integral of (7) is zero for positive values of z', whereas in the first case we may take h_2 instead of z' for the upper limit. Integration by parts gives

$$\int_{-h_1}^{h_2} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{i\frac{2\pi z'}{TV}} dz' = \Big/_{-h_1}^{h_2} \frac{\partial \mathfrak{W}_{x'}}{\partial z'} e^{i\frac{2\pi z'}{TV}} - i\frac{2\pi}{TV} \int_{-h_1}^{h_2} \frac{\partial \mathfrak{W}_{x'}}{\partial z'} e^{i\frac{2\pi z'}{TV}} dz'.$$

Now $\mathfrak{E}^e$ can only be represented by (1) if it is a *continuous* function of the co-ordinates, but we may imagine nevertheless that at the boundary of the space τ, $\mathfrak{W}$ and $\partial \mathfrak{W}/\partial z'$ have arbitrarily small values. These quantities may therefore be taken zero at the boundary; as to $\mathfrak{W}$, this has already been done in the considerations of the preceding paragraph. Hence the first term, given by the integration by parts, vanishes; the second may again be integrated by parts, so that finally

$$\int_{-h_1}^{h_2} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{i\frac{2\pi z'}{TV}} dz' = -\frac{4\pi^2}{T^2V^2} \int_{-h_1}^{h_2} \mathfrak{W}_{x'} e^{i\frac{2\pi z'}{TV}} dz'.$$

The exponential factor under the sign of integration may be replaced by 1. Indeed, if a certain length l, of the same order of magnitude as the linear dimensions of the space τ is very small in comparison with the wavelength λ of light, we may omit terms containing products of $\frac{\tau}{\lambda^3}$ and quantities of the order $\frac{l}{\lambda}$. Now

$$\mathfrak{W}_{x'} = \int \mathfrak{E}^e_{x'}\, d\sigma$$

the integral taken over the portion of a plane $z' =$ const. lying within τ. From this we infer

$$\int_{-h_1}^{h_2} \mathfrak{W}_{x'}\, dz' = \int \mathfrak{E}^e_{x'}\, d\tau$$

integrated over the volume τ. We shall represent this integral by

$\mathfrak{E}^e_{x'}\tau$, denoting by $\mathfrak{E}^e_{x'}$ a certain mean value of the X'-component of the E. M. F. within τ. We may now write

$$\int_{-h_1}^{h_2} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{i\frac{2\pi z'}{TV}} dz' = -\frac{4\pi^2 \mathfrak{E}^e_{x'}\tau}{T^2V^2}$$

Similarly

$$\int_{-h_1}^{h_2} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{-i\frac{2\pi z'}{TV}} dz' = -\frac{4\pi^2 \mathfrak{E}^e_{x'}\tau}{T^2V^2},$$

an integral that has to be used for negative values of z' less than $-h_1$.

§ 6. If lastly

$$-h_1 < z' < h_2$$

the point P lies between the tangent planes and *both* integrals differ from zero. We find by some transformations

$$\int_{-h_1}^{z'} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{i\frac{2\pi z'}{TV}} dz' = \frac{\partial \mathfrak{W}_{x'}}{\partial z'} e^{i\frac{2\pi z'}{TV}} - i\frac{2\pi}{TV}\mathfrak{W}_{x'} e^{i\frac{2\pi z'}{TV}} - \frac{4\pi^2}{T^2V^2}\int_{-h_1}^{z'} \mathfrak{W}_{x'} e^{i\frac{2\pi z'}{TV}} dz'$$

$$\int_{h_2}^{z'} \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2} e^{-i\frac{2\pi z'}{TV}} dz' = \frac{\partial \mathfrak{W}_{x'}}{\partial z'} e^{-i\frac{2\pi z'}{TV}} + i\frac{2\pi}{TV}\mathfrak{W}_{x'} e^{-i\frac{2\pi z'}{TV}} -$$

$$- \frac{4\pi^2}{T^2V^2}\int_{h_2}^{z'} \mathfrak{W}_{x'} e^{-i\frac{2\pi z'}{TV}} dz'.$$

So that in this case the X'-component of the electric force is given by

$$\mathfrak{E}_{x'} = \frac{i\,d\omega}{8\pi TV}\Bigg[-i\frac{4\pi}{TV}\mathfrak{W}_{x'} - \frac{4\pi^2}{T^2V^2} e^{-i\frac{2\pi z'}{TV}} \int_{-h_1}^{z'} \mathfrak{W}_{x'} e^{i\frac{2\pi z'}{TV}} dz' -$$

$$- \frac{4\pi^2}{T^2V^2} e^{i\frac{2\pi z'}{TV}} \int_{z'}^{h_2} \mathfrak{W}_{x'} e^{-i\frac{2\pi z'}{TV}} dz'\Bigg].$$

Since z' lies between $-h_1$ and h_2, we may replace the exponential factors by 1, both before and behind the sign of integration. Then we find finally

1st. If P lies between the tangent planes

$$\mathfrak{E}_{x'} = \frac{\mathfrak{W}_{x'}}{2T^2V^2} d\omega - \frac{i\pi \mathfrak{E}^e_{x'}\tau}{2T^3V^3} d\omega$$

2nd. If P lies outside these planes

a. For positive values of z'

$$\mathfrak{E}_{x'} = -\frac{i\pi\mathfrak{E}^e_{x'}\tau}{2T^3V^3} e^{-i\frac{2\pi z'}{TV}} d\omega$$

b. For negative values of z'

$$\mathfrak{E}_{x}' = -\frac{i\,\pi\,\mathfrak{E}^e_{x'}\,\tau}{2\,T^3V^3} e^{i\frac{2\,\pi z'}{T\,V}} d\omega.$$

The Z'-component of the electric force consists of two parts, one of which corresponds to $\mathfrak{E}_{x'}$, the other to $\mathfrak{E}_{y'}$. Having already omitted the Y'-component, we shall take only the first part, $\mathfrak{E}_{z'1}$, of the Z'-component and add the second part $\mathfrak{E}_{z'2}$ to the Y'-component afterwards. Then by the third equation of (4)

$$\mathfrak{E}_{z'1} - \frac{1}{8\pi^2}\frac{\partial^2\mathfrak{W}_{z'}}{\partial z'^2} d\omega = -\frac{\varepsilon_{13}}{\varepsilon_{33}}\left\{\mathfrak{E}_{x'} - \frac{1}{8\pi^2}\frac{\partial^2\mathfrak{W}_{x'}}{\partial z'^2} d\omega\right\}.$$

It appears from this that outside the tangent planes $\mathfrak{E}_{z'}$ and $\mathfrak{E}_{x'}$ are connected with each other in the way they always are in the case of plane waves. We may therefore represent the electric force by $\frac{\mathfrak{E}_{x'}}{\cos\vartheta}$ if ϑ is the angle between this vector and the corresponding dielectric displacement in a system of plane waves. Finally we have the following equations for the components of the electric force along the axes of symmetry

$$\left.\begin{aligned} \mathfrak{E}_x &= -\frac{i\pi\alpha\mathfrak{E}^e_{x'}\tau}{2T^3V^3\cos\vartheta} e^{-i\frac{2\pi z'}{TV}} d\omega, \\ \mathfrak{E}_y &= -\frac{i\pi\beta\mathfrak{E}^e_{x'}\tau}{2T^3V^3\cos\vartheta} e^{-i\frac{2\pi z'}{TV}} d\omega, \\ \mathfrak{E}_z &= -\frac{i\pi\gamma\mathfrak{E}^e_{x'}\tau}{2T^3V^3\cos\vartheta} e^{-i\frac{2\pi z'}{TV}} d\omega, \end{aligned}\right\} \quad \ldots\ldots \quad (8)$$

where α, β and γ are the direction cosines of the electric force with respect to the axes of symmetry. For negative values of z' the same formulae will apply, provided that z' be replaced by $-z'$.

§ 7. In the preceding equations the symbols $\mathfrak{E}_x$, $\mathfrak{E}_y$ and $\mathfrak{E}_z$ were used for the (small) electric force, produced by a single element of the integral (1) in a point P, lying at a given distance r from the origin O. We have seen that the expression for this small electric force took a different form according to the point P lying or not lying between the before mentioned tangent planes. Now the directions for which P lies *inside* these planes are those *excluded* by a certain cone K, which may be defined as the locus of all normals to another cone, having its vertex in P and tangent to the boundary of the element of space τ. On the other hand, all directions of wave normals, for which P lies *outside* the tangent planes are *included* by the cone K. It is clear that this cone will differ infinitely little from the plane passing through O perpendicularly to OP.

We may therefore find the total electric force by integrating the right hand members of the equations (8) with respect to all directions lying within K and then adding to the result the quantity obtained by integrating the expressions relating to the remaining directions. In effecting the first integration we must replace z' by $-z'$ for negative values of z', according to the remark made at the end of § 6. But we may as well limit the integration to half the cone K multiplying the result by 2. Again, we may extend this integration to the plane F; indeed the right hand members of (8) contain τ as a factor, so that it does not matter, whether or no an infinitely small solid angle is included in this integration.

It remains to consider the expressions

$$\mathfrak{E}_{x'} = \frac{\mathfrak{W}_{x'}}{2T^2V^2}\,d\omega - \frac{i\pi\,\mathfrak{E}_{x'}\,\tau}{2T^3V^3}\,d\omega,$$

$$\mathfrak{E}_{z'1} = \frac{1}{8\pi^2}\left(\frac{\partial^2\mathfrak{W}_{z'}}{\partial z'^2}\,d\omega + \frac{\varepsilon_{12}}{\varepsilon_{33}}\frac{\partial^2\mathfrak{W}_{x'}}{\partial z'^2}\,d\omega\right) - \frac{\varepsilon_{13}}{\varepsilon_{33}}\,\mathfrak{E}_{x'},$$

which have to be integrated over all directions outside the cone K.

Now, from these expressions we get the components along the axes of symmetry by multiplying them by finite factors. It is easily seen that terms already containing the factor τ may therefore be omitted, so that we may write

$$\mathfrak{E}_{x'} = \frac{\mathfrak{W}_{x'}}{2T^2V^2}\,d\omega;$$

$$\mathfrak{E}_{z'1} = -\frac{\varepsilon_{13}}{\varepsilon_{33}}\,\mathfrak{E}_{x'} + \frac{1}{8\pi^2}\left(\frac{\partial^2\mathfrak{W}_{z'}}{\partial z'^2} + \frac{\varepsilon_{13}}{\varepsilon_{33}}\frac{\partial^2\mathfrak{W}_{x'}}{\partial z'^2}\right)d\omega.$$

§ 8. We shall resolve this last vector into two other vectors, the components of the first being

$$\mathfrak{E}_{x'} = \frac{\mathfrak{W}_{x'}}{2T^2V^2}\quad,\quad \mathfrak{E}_{z'1} = -\frac{\varepsilon_{13}}{\varepsilon_{33}}\,\mathfrak{E}_{x'},$$

and those of the second

$$\mathfrak{E}_{x'} = 0, \qquad \mathfrak{E}_{z'1} = \frac{1}{8\pi^2}\left(\frac{\partial^2\mathfrak{W}_{z'}}{\partial z'^2} + \frac{\varepsilon_{13}}{\varepsilon_{33}}\frac{\partial^2\mathfrak{W}_{x'}}{\partial z'^2}\right)d\omega.$$

The first vector has again the same direction as the electric force in plane waves whose normal coincides with the direction we are considering; its components along the axes of symmetry are therefore

$$\mathfrak{E}_x = \frac{\alpha \mathfrak{W}_{x'}}{2\, T^2 V^2 \cos\vartheta} d\omega\ , \quad \mathfrak{E}_y = \frac{\beta \mathfrak{W}_{x'}}{2\, T^2 V^2 \cos\vartheta} d\omega\ , \quad \mathfrak{E}_z = \frac{\gamma \mathfrak{W}_{x'}}{2 T^2 V^2 \cos\vartheta} d\omega.$$

Now $\mathfrak{W}_{x'}$, is of the order l^2 and the integration is to be effected over a solid angle of the order l. Thus, confining ourselves to directions in a single plane passing through OP, we may regard as constants the quantities α, V and $\cos\vartheta$, assigning to them the values they take in the plane F.

We determine an arbitrary direction in the plane passing through OP by the angle ζ which it makes with OP and its azimuth χ with respect to, a fixed plane also passing through OP. Then

$$d\omega = \sin\zeta\, d\zeta\, d\chi.$$

Now we have for the direction considered

$$W_{x'} = \int \mathfrak{E}^e_{x'}\, d\sigma$$

the integral being extended to the portion inside τ of a plane G, passing through P perpendicularly to that direction. If q is the normal drawn from O towards G, we have

$$q = r \cos\zeta,$$
$$|dq| = r \sin\zeta\, d\zeta,$$

giving

$$d\omega = \frac{1}{r}\, |dq|\, d\chi,$$

and

$$\mathfrak{E}_x = \frac{1}{2rT^2}\int_0^{2\pi} \frac{\alpha}{V^2 \cos\vartheta} d\chi \int \mathfrak{W}_{x'}\, |dq|.$$

Here for each particular value of χ, the latter integral is to be extended to all values that can be given to ζ or q. Further

$$\int \mathfrak{W}_{x'}\, |dq| = \int |dq| \int \mathfrak{E}^e_{x'}\, d\sigma = \int\int \mathfrak{E}^e_{x'}\, d\sigma\, |dq|,$$

whereas

$$d\sigma\, |dq|.$$

is the element of volume of an infinitely small cylinder whose upper and lower base are formed respectively by one of the surface elements of G and of an infinitely near plane G', the generating lines of the cylinder being perpendicular to G. It follows from this that

$$\iint \mathfrak{E}^e_{x'}\, d\sigma\, |dq|$$

is the volume-integral of $\mathfrak{E}^e_{x'}$, taken over the whole volume of τ. We have already written for this integral $\mathfrak{E}^e_{x'}\,\bar{\tau}$, denoting by $\mathfrak{E}^e_{x'}$ a certain mean value (§ 5). Hence, the *first* part of the components of the electric force resulting from the integration with respect to the directions outside the cone K, becomes

$$\mathfrak{E}_x = \frac{1}{2T^2 r}\int_0^{2\pi} \frac{\alpha \mathfrak{E}_x\, \tau}{V^2 \cos\vartheta}\, d\chi\ ,\ \mathfrak{E}_y = \frac{1}{2T^2 r}\int_0^{2\pi} \frac{\beta \mathfrak{E}_{x'}\, \tau}{V^2 \cos\vartheta}\, d\chi\ ,\ \mathfrak{E}_z =$$

$$= \frac{1}{2T^2 r}\int_0^{2\pi} \frac{\gamma \mathfrak{E}_{x'}\, \tau}{V^2 \cos\vartheta}\, d\chi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The second part results from a similar integration of the second vector

$$\mathfrak{E}_{x'} = 0\ ,\ \mathfrak{E}_{z'1} = \frac{1}{8\pi^2}\left(\frac{\partial^2 \mathfrak{W}_{z'}}{\partial z'^2} + \frac{\varepsilon_{13}}{\varepsilon_{33}}\,\frac{\partial^2 \mathfrak{W}_{z'}}{\partial z'^2}\right) d\omega.$$

Now it will appear further on, that we can only determine the exact value of those terms, in which the denominator contains the first power of r. We may therefore confine ourselves to such terms in the whole course of our calculations. The cone over which we have to integrate being of the order l/r, we may omit terms, which already contain r in the denominator. It will be evident therefore that instead of

$$\frac{\partial^2 \mathfrak{W}_{z'}}{\partial z'^2} \text{ and } \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2}$$

we may take the values of these quantities, corresponding to that wave-normal, in the meridian plane passing through OP, which lies at the same time in the plane F. If dz' is a line-element of that wave-normal, we have to consider the integrals

$$\int \frac{\partial^2 \mathfrak{W}_{z'}}{\partial z'^2}\, dz' \text{ and } \int \frac{\partial^2 \mathfrak{W}_{x'}}{\partial z'^2}\, dz'$$

which evidently are zero, $\frac{\partial \mathfrak{W}}{\partial z'}$ being zero at the boundary of τ. It appears in this way that we need not at all consider the second vector.

§ 9. We now proceed to effect the integration of the right hand members of the equations (8) so far as is necessary in order to obtain the terms with $\frac{1}{r}$. We shall take the real parts of all expressions and represent henceforth by $\mathfrak{E}$ the whole electric force. Then, if $\mathfrak{E}^e = \mathfrak{b} \cos 2\pi \frac{t}{T}$, we shall have

$$\mathfrak{E}_x = \int \frac{\pi \, a \, \mathfrak{b}_{x'} \, \tau}{T^3 \, V^3 \cos \vartheta} \sin \frac{2\pi}{T} \left(t - \frac{z'}{V} \right) d\omega, \quad . \quad . \quad . \quad (10)$$

integrated over all directions on that side of F where z' has positive values. We therefore obtain the resultant luminous vibration in an arbitrary point P as the sum of small vibrations, belonging to a great number of systems of plane waves of all possible directions. These vibrations differ from each other in amplitude and in phase. The changes of phase are determined by those of the quantity

$$\frac{z'}{TV}.$$

Since TV means the wave-length in the crystal for the direction considered and $z' = r \cos \zeta$, the phase will vary very much by small variations of ζ, i.e., of the direction of the wave system in question. There is one direction for which

$$\frac{z'}{TV}$$

takes a maximum value. This is the direction of the wave-normal OQ to which OP corresponds as first ray. Indeed, z'/TV is proportional to the time in which the vibrations of a certain wave-system arrive at P and this time is really a maximum for the system whose normal is OQ. We shall prove, that the resultant vibration at P is the same as it would be, if we had only to do with wave systems of this latter direction and of directions in the immediate vicinity of it. To this effect we shall fix our attention on an arbitrary normal ON, making an angle ϕ with OQ, writing ψ for the azimuth of the plane NOQ with respect to a fixed plane, which passes through OQ, and for which we might take the plane POQ. We shall not however introduce ψ and ϕ as variables but ψ and

$$u = \frac{V_0}{V} \cos \zeta,$$

if V_0 is the velocity of propagation of the plane wave, having OQ for its normal. Further we put

$$\frac{2\pi t}{T} = h, \quad \frac{2\pi r}{TV_0} = g, \quad d\omega = \sin \phi \frac{\partial \phi}{du} \, du \, d\psi .$$

Then

$$\mathfrak{E}_x = - \int_0^{2\pi} \int_{u_0}^{0} \frac{\pi a \mathfrak{b}_{x'} \tau}{T^3 V^3 \cos \vartheta} \sin (gu - h) \sin \phi \frac{\partial \phi}{du} \, du \, d\psi , \quad . \quad . \quad . \quad (11)$$

if u_0 is the value of u for the direction OQ. Indeed the directions for which $u = \text{const.}$ lie on a cone surrounding the line OQ, just because u is a maximum for that line. We first integrate with respect to ψ and put

$$-\int_0^{2\pi} \frac{\pi \alpha \mathfrak{b}_{x'} \tau}{T^3 V^3 \cos\vartheta} \sin\phi \frac{\partial \phi}{du} d\psi = f(u) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The result is

$$\mathfrak{E}_x = \int_{u_0}^{0} f(u) \sin(gu - h)\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

§ 10. An integral such as (13) has already been considered by KIRCHHOFF. For great values of g it approaches uniformly to zero and at infinity it may be represented by a development of the form

$$\frac{a_1}{g} + \frac{a_2}{g^2} + \quad . \quad . \quad .$$

It is only the coefficient a_1 that can be found. Integration by parts of the integral gives

$$\int_{u_0}^{0} f(u) \sin(gu - h)\, du = \frac{f(u_0) \cos(gu_0 - h) - f(0) \cos h}{g} + \frac{a_2}{g^2} + \ldots \quad (14)$$

The first term, taken by itself, gives a sufficiently exact result for points P, lying at distances r from O, which are large in comparison with the wavelength of light; in the following development we have in view only such points as satisfy this condition. We put therefore

$$\int_{u_0}^{0} f(u) \sin(gu - h)\, du = \frac{f(u_0) \cos(gu_0 - h) - f(o) \cos h}{g} \quad . \quad . \quad (14^a)$$

We shall first consider the part

$$-\frac{f(0)\cos h}{g} = \frac{T V_0}{2\pi r} \cos 2\pi \frac{t}{T} \int_0^{2\pi} \left[\frac{\pi \alpha\, b_{x'} \tau}{T^3\, V^3 \cos\vartheta} \sin\phi \frac{\partial \phi}{\partial u} \right]_{u=0} d\psi.$$

Now

$$Sin\, \phi \frac{\partial \phi}{\partial u} = -\frac{\partial(\cos\phi)}{\partial(\cos\zeta)} \cdot \frac{\partial(\cos\zeta)}{\partial u},$$

$$\frac{\partial u}{\partial(\cos\zeta)} = \frac{\partial}{\partial(\cos\zeta)} \left[\frac{V_0}{V} \cos\zeta \right] = \frac{V_0}{V} + \cos\zeta \frac{\partial}{\partial(\cos\zeta)} \left[\frac{V_0}{V} \right],$$

so that for $u = 0$ or $\cos\zeta = 0$

$$\left[\sin\Phi\frac{\partial\Phi}{\partial u}\right]_{u=0}=-\frac{V}{V_0}\left[\frac{\partial(\cos\Phi)}{\partial(\cos\zeta)}\right]_{u=0}.$$

We may further deduce from the consideration of the spherical triangle, defined by the directions ON, OQ and OP, that for $u=0$

$$\frac{\partial(\cos\Phi)}{\partial(\cos\zeta)}=\left(\frac{d\chi}{d\psi}\right)_{u=0},$$

so that

$$\left(\sin\Phi\frac{\partial\Phi}{\partial u}\right)_{u=0}=-\frac{V}{V_0}\left(\frac{d\chi}{d\psi}\right)_{u=0}$$

and

$$-\frac{f(0)\cosh}{g}=-\frac{1}{2T^2r}\cos\frac{2\pi t}{T}\int_0^{2\pi}\frac{a\mathfrak{b}_{x'}\tau}{V^2\cos\vartheta}d\chi\,.$$

The real part of the expression (9), added to this result gives exactly zero, so that, as we could have expected, there remains in $\mathfrak{E}_x$ no term with only $\cos 2\pi t/T$. We need hardly add that this is equally the case with $\mathfrak{E}_y$ and $\mathfrak{E}_z$.

Finally we have to determine $f(u_0)$. Let us denote by Ω the solid angle of a cone, formed by directions for which u is constant, then

$$d\Omega=du\int_0^{2\pi}\sin\Phi\frac{\partial\Phi}{\partial u}d\psi\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Now by (12) we have

$$f(u_0)=-\frac{\pi a_0\mathfrak{b}_{x_0'}\tau}{T^3V_0^{\,3}\cos\vartheta_0}\int_0^{2\pi}\left(\sin\Phi\frac{\partial\Phi}{\partial u}\right)_{u=u_0}d\psi\,,$$

and with a view to (15) we may write for this

$$f(u_0)=-\frac{\pi a_0\mathfrak{b}_{x_0'}\tau}{T^3V_0^{\,3}\cos\vartheta_0}\left(\frac{d\Omega}{du}\right)_{u=u_0}.$$

The solid angle $d\Omega_0$ of an infinitely small cone with axis OQ may be found in the following manner. We imagine the wave-surface W, passing through P, and the polar surface R of W with respect to a sphere of radius unity. Then the point corresponding to P will be the point of intersection Q of OQ and R. Further we take a point P' on OP prolonged, close to P and describe from P' the cone tangent to W. The normals drawn from O to this cone will lie on a second cone and this is the locus of all directions for which u has the constant value

$$\frac{OP}{OP'} \cos \vartheta_0$$

The infinitely small cone of normals will intersect R in a curve lying in a plane, normal to OP; the plane touching R at the point Q is also normal to OP. Let these last two planes, which are therefore parallel, cut OP in S' and S. Then

$$OS \times OP = 1 \quad OS' \times OP' = 1\,,$$

and

$$u = OS' \,.\, OP \cos \vartheta_0$$
$$du_0 = -SS' \,.\, OP \cos \vartheta_0 \,.$$

Further

$$d\Omega_0 = \frac{\cos \vartheta_0}{OQ^2}\, d\sigma\,,$$

if $d\sigma$ is the infinitely small surface of the just mentioned plane curve. But we have also

$$d\sigma = 2\pi \sqrt{\varrho_0\, \varrho'_0}\, SS'$$

if ϱ_0 and ϱ'_0 are the two principal radii of curvature of R at the point Q. Combining the obtained equations we find therefore

$$\left(\frac{d\Omega}{du}\right)_{u=u_0} = -\frac{2\pi \sqrt{\varrho_0\, \varrho'_0}}{OQ^2\, OP}\,,$$

or since $OP = r$ and $\dfrac{1}{OQ} = OP \cos \vartheta_0 = r \cos \vartheta_0$

$$\left(\frac{d\Omega}{du}\right)_{u=u_0} = -2\pi r \sqrt{\varrho_0\, \varrho'_0} \cos^2 \vartheta_0\,,$$

so that

$$f(u_0)\,\frac{\cos(gu_0 - h)}{g} = -\frac{\pi \alpha_0\, \mathfrak{b}_{x_0}\, \tau}{T^3 V_0^{\,3} \cos \vartheta_0} \,.\, -2\pi r \sqrt{\varrho_0 \varrho'_0} \cos^2 \vartheta_0 \,.\, \frac{T V_0}{2\pi r} \cos(gu_0 - h)$$

or by (13) and (14)

$$\mathfrak{E}_x = f(u_0)\frac{\cos(gu_0 - h)}{g} = \frac{\pi \alpha_0\, \mathfrak{b}_{x'_0}\, \tau \sqrt{\varrho_0\, \varrho'_0} \cos \vartheta_0}{T^2\, V_0^{\,2}} \cos \frac{2\pi}{T}\left(t - \frac{r}{p_0}\right),$$

if p_0 is the velocity of propagation of the *ray* OP, as it is defined for *plane* waves. Thus the electric force appears to have the same direction as it has for *plane* waves whose corresponding rays coïncide with OP. Its magnitude is given by

$$\mathfrak{E} = \frac{\pi\, \mathfrak{b}_{x_0'}\, \tau \sqrt{\varrho_0\, \varrho_0'} \cos \vartheta_0}{T^2\, V_{;0}^{\,2}} \cos \frac{2\pi}{T}\left(t - \frac{r}{p_0}\right).$$

§ 11. We must add to this a second vibration which may be obtained by the composition of all wave systems due to the Y'-components of the infinitely small vectors into which the original E. M. F. has been divided. It is this action we have left aside in

§ 3; the total electric force produced by it is given by

$$\mathfrak{E} = \frac{\pi\, \mathfrak{b}_{y_1}'\, \tau \sqrt{\varrho_1\, \varrho_1'}\, \cos\vartheta_1}{T^2\, V_1^{\,2}} \cos\frac{2\pi}{T}\left(t - \frac{r}{p_1}\right),$$

if we distinguish by the index 1 the quantities corresponding to the *second* plane wave for which OP is the direction of the ray. The magnetic force too has in both cases the ordinary direction and may be derived from the electric force by multiplying respectively by

$$\frac{c}{p_0} = \frac{c \cos\vartheta_0}{V_0} \quad \text{and} \quad \frac{c}{p_1} = \frac{c \cos\vartheta_1}{V_1},$$

so that the flow of energy is given by

$$\mathfrak{S} = c.\frac{c \cos\vartheta_0}{V_0}\, \mathfrak{E}_0^{\,2} + c.\frac{c \cos\vartheta_1}{V_1}\, \mathfrak{E}_1^{\,2},$$

or

$$\mathfrak{S} = \frac{\pi^2 c^2\, \mathfrak{b}^2_{x'_0}\, \tau^2\, \varrho_0\, \varrho'_0 \cos^3\vartheta_0}{T^4\, V_0^{\,5}} \cos^2\frac{2\pi}{T}\left(t - \frac{r}{p_0}\right) + \\ + \frac{\pi^2 c^2\, \mathfrak{b}^2_{y'_1}\, \tau^2\, \varrho_1\, \varrho'_1 \cos^3\vartheta_1}{T^4\, V_1^{\,5}} \cos^2\frac{2\pi}{T}\left(t - \frac{r}{p_1}\right).$$

The mean flow of energy per unit time is therefore

$$\mathfrak{S} = \frac{\pi^2 c^2}{2T^4}\left[\frac{\mathfrak{b}^2_{x'_0}\, \tau^2\, \varrho_0\, \varrho'_0 \cos^3\vartheta_0}{V_0^{\,5}} + \frac{\mathfrak{b}^2_{y'_0}\, \tau^2\, \varrho_1\, \varrho'_1 \cos^3\vartheta_1}{V_1^{\,5}}\right].$$

The amount of energy travelling outwards in directions lying within the cone of rays do', is

$$r^2\, \mathfrak{S}\, do'.$$

We may finally observe that the cone of corresponding wave-normals has a solid angle

$$do = \varrho\, \varrho' \cos^3\vartheta \,.\, r^2\, do'$$

so that the total amount of energy radiating from the centre may be represented by the integral

$$E = \frac{\pi^2 c^2}{2T^4} \int \left(\frac{\mathfrak{b}^2_{x'}\, \tau^2}{V_{x'}^{\,5}} + \frac{\mathfrak{b}^2_{y'}\, \tau^2}{V_{y'}^{\,5}}\right) do.$$

It is only in the case of uniaxial crystals that this integral can be further calculated.

Geology. — *"On brackish and fresh water deposits of the river Silat in Western-Borneo."* By Prof. K. Martin.

(This communication will not be published in these Proceedings).

(March 22, 1906).