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The following pages contain a condensed summary of the results of an investigation, which will soon be published in detail in the "*Annals of the Royal Observatory at the Cape of Good Hope*". The material on which this investigation is based consists entirely of observations made at the Cape Observatory, viz.:

1. Heliometer-observations made in 1891 by GILL and FINLAY, discussed by me and published in my inaugural dissertation. [1]
2. Photographic plates taken at the Cape Observatory in 1891, measured and discussed by me.
3. Heliometer-observations made in 1901 and 1902 by COOKSON, discussed by himself and published in Monthly Notices, June 1904 p. 728—747.
4. Photographic plates taken in 1903 and 1904, measured and discussed by me.

[1]) Discussion of Heliometer-Observations of Jupiter's Satellites, Groningen J. B. WOLTERS 1901.

My aim with this investigation was exclusively the determination of the inclinations and nodes of the orbital planes of the satellites and of the motions of these nodes. The plates of 1903 and 1904 were taken in order to provide a second epoch from which these motions could be determined by a comparison with the observations of 1891.

The fine series of observations, made by Mr. BRYAN COOKSON in 1901 and 1902 increases the weight of this determination considerably.

I have already pointed out, in the fourth chapter of my dissertation, that the determination of the other elements, which must be derived from the observed (jovicentric) longitudes, is probably sufficiently provided for by the observations of eclipses. Moreover from the observations mentioned above sub 1. and 3. *all* elements were determined.

Eclipse-observations are however not well adapted for the determination of the inclinations and nodes, which must be derived from the observed latitudes, as I have shown, l. c. page 77. The principal interest of the determination of the orbital planes lies in the comparison with the observations of the large motions of the nodes, which are demanded by the theory. Since these motions are produced almost exclusively by the large polar compression of the planet, the natural fundamental plane to which the latitudes must be referred is the equator of Jupiter.

If we refer the positions of the satellites to a system of co-ordinate axes, of which the axis of y is the projection on the sphere of a line perpendicular to this fundamental plane (i. e. of Jupiter's axis of rotation), and the axis of x is the great circle through the centre of the planet perpendicular to the axis of y, then for the determination of the inclinations and nodes the y co-ordinates of the satellites are alone important. Only these co-ordinates have therefore been measured. The plate was, by means of the position-circle with which the Repsold measuring machine of the Astronomical Laboratory at Groningen is provided, brought approximately in the position-angle $P + 90°$, where P is the position-angle of Jupiter's adopted axis of rotation. The plate then has a motion parallel to a straight line, which nearly coincides with the axis of x, and which is defined by the axis of the cylinder which guides the plate-holder in its motion. The co-ordinates perpendicular to this straight line were then measured by the micrometer screw. These differ from the co-ordinates y only by small corrections (refraction, orientation and scale-value). In this method the measured quantities never exceed a few revolutions of the screw. All errors of réseau-lines, division errors of the scales, error of projection, etc. are avoided. The straightness of the cylinder was repeatedly tested by comparison with a stretched spiderline. Its errors are certainly smaller than 0.2 micron. The position-angles were read off by two microscopes, and the orientation of the plate was determined from a pair of standard stars, which were for this purpose photographed on each plate, and from trails of the satellites. The errors of observation of the measures of the satellites are satisfactory, distortion of the photographic film cannot be detected, and the discussion of a dozen plates, which were specially taken for this purpose, shows that the determination of the orientation from the trails is always practically free from systematic errors, while the same can be said of the determination from the standard stars under certain conditions, which are however not always fulfilled. The accidental errors of both determinations are very small.

The image of the planet has not been measured. The observed co-ordinates contain therefore an unknown additive constant (different for each separate plate), which was eliminated by using in the subsequent reductions the co-ordinates referred to the mean of all the satellites occurring on the plate as origin. The equations of condition and normal equations for these relative co-ordinates are very simple and symmetrical. The limited space at my disposal prevents me however from entering into more details regarding the measures and reductions. I will at once state the results.

The unknowns which were determined from each opposition were the corrections to the adopted values of the elements p and q of the four satellites which are defined by the formulas:

$$p = i \sin(-\Omega)$$
$$q = i \cos(-\Omega)$$

where i and Ω are the inclination and ascending node of the orbital plane of the satellite referred to the fundamental plane. The longitude of the node is counted from the ascending node of the plane of Jupiter's orbit on the fundamental plane. The quantities referring to the four satellites are distinguished by the suffixed numerals 1 to 4.

The following table contains the results of the different series of observations with their probable errors.

The values for 1891 (*Heliometer*) are those derived in my dissertation with a few unsignificant corrections in the last decimal places. The results from the heliometer and those from the plates have been combined with the relative weights 2 and 1.

The results for 1901 and 1902 are quoted from the communication by Mr. Cookson in the Monthly Notices.

I have however been compelled to reject Δp_1 and Δq_1 for 1901

	1891 75 Heliometer.	1891.75 Plates.	1891.75
Δp_1	+ 0°.0409 ± °.0052	+ 0°.0372 ± °.0050	+ 0°.0397 ± °.0038
Δp_2	+ .0740 ± 36	+ .0733 ± 36	+ .0738 ± 27
Δp_3	— .0033 ± 23	— .0024 ± 19	— .0030 ± 17
Δp_4	+ .0642 ± 12	+ .0638 ± 12	+ .0641 ± 9
Δq_1	— 0.0259 ± .0056	— 0.0258 ± .0061	— 0.0259 ± .0043
Δq_2	+ 0853 ± 33	+ .0813 ± 35	+ .0840 ± 25
Δq_3	— .0683 ± 20	— .0748 ± 23	— .0705 ± 16
Δq_4	— .0137 ± 12	— .0117 ± 11	— .0130 ± 9

	1901.61 COOKSON.	1902 62 COOKSON.	1903.72 Plates.	1904.89 Plates.
Δp_1	+ 0°.0170 ± °.0077	+ 0°.0137 ± °.0072	+ 0°.0021 ± °.0060	— 0°.0028 ± °.0078
Δp_2	+ .1113 ± 56	+ .0922 ± 44	+ .0526 ± 33	+ .0158 ± 44
Δp_3	— .0148 ± 36	— .0072 ± 28	— .0199 ± 22	— .0104 ± 28
Δp_4	+ .0456 ± 18	+ .0658 ± 15	+ .0637 ± 12	+ .0648 ± 13
Δq_1	— 0.0695 ± .0084	— 0.0755 ± .0065	— 0.0597 ± .0048	— 0.0335 ± .0077
Δq_2	— .1770 ± 49	— .1860 ± 40	— .2120 ± 32	— .2252 ± 48
Δq_3	— .0360 ± 33	— .0334 ± 23	— .0494 ± 20	— .0476 ± 26
Δq_4	— .0359 ± 18	— .0070 ± 14	— .0284 ± 11	— .0209 ± 17

and 1902. COOKSON found from the reduction of his observations that the residuals could be much reduced by assuming in the latitude of satellite IV an inequality of which the period is one half of the periodic time of the satellite while the coefficient is about 50″. I have searched for this inequality in the observations of 1891, 1903 and 1904 and I can confidently declare that in none of these years there is even the slightest trace of any inequality of which the argument should be a multiple of the mean longitude of Sat. IV. Since also an inequality of this nature cannot be explained by the theory I cannot but doubt its reality, and since the cause which has produced this apparent inequality must necessarily also have affected the determination of p_4 and q_4, the safest course seemed to be to reject the values of these elements found from the observations of 1901 and 1902. All other corrections Δp and Δq derived from the observations are included in the following discussion, with weights inversely proportional to the squares of their probable errors and corresponding to a p. e. of weight unity of $\pm 0.^\circ 0050$.

Before this discussion can be related the theoretical expressions for p and q must be developed.

At the time when the analytical theory of the satellites was created by LAGRANGE and LAPLACE, the eclipses were practically the only phenomena of the satellites which were observed. For these the natural fundamental plane is the plane containing the axis of the shadow-cone, i. e. the plane of Jupiter's orbit. This was accordingly used by them. SOUILLART, in his theory published in 1880, followed their example.

The first thing which must be done before the theory can be compared with modern observations is thus to reduce the expressions for the latitudes referred to Jupiter's orbit to latitudes referred to the equator. This has already been done by MARTH, who in 1891 published tables for the computation of the co-ordinates of the satellites, based on SOUILLART's theory (Monthly Notices, June 1891, pages 505—539).

Let I and N be the inclination and node [1]) of the orbital plane of one of the satellites with reference to the orbit of Jupiter. SOUILLART's theory then gives

$$\left.\begin{aligned} I_i \sin N_i &= \sum_{j=1}^{4} b_{ij} \sin \theta_j + \mu_i\, \omega \sin \theta_0 \\ I_i \cos N_i &= \sum_{j=1}^{4} b_{ij} \sin \theta_j + \mu_i\, \omega \sin \theta_0 \end{aligned}\right\} \quad . \;.\; .\; . \quad (1)$$

In these formulae ω and θ_0 are the inclination and node of Jupiter's equator on its orbit. All longitudes are counted from the first point of Aries. The quantities b_{ij} are constants, and the angles θ_i vary proportionally with the time. Of the constants b_{ij} four only are mutually independent. If we put:

$$b_{ii} = \gamma_i \qquad b_{ij} = \sigma_{ij}\, \gamma_j \;,$$

then the γ_i are constants. The multipliers σ_{ij} and μ_i and the coefficients of the time in the expressions for θ_i are given by the theory as functions of the masses, the compression of Jupiter and the mean motions. The constants σ_{ij} are small numbers (the largest is $\sigma_{43} = 0.1944$) with the exception, of course, of those in the diagonal, $\sigma_{ii} = 1$. The value of μ_i differs little from unity. The angles γ_i and θ_i are what LAPLACE calls the "inclinaisons et noeuds propres" of the satellites.

[1]) With node I mean ascending node, unless otherwise stated.

Let now ω_0 and ψ_0 be the inclination and the longitude (counted from the first point of Aries) of the descending node of the plane which I wish to adopt as the fundamental plane, referred to the plane of Jupiter's orbit. Longitudes in the fundumental plane are counted from the node ψ_0 as zero.

Then if i and Ω are the inclination and node of the orbit of one of the satellites referred to the fundamental plane, we have, neglecting quantities of the third order in i, I and ω_0:

$$i \sin \Omega = I \sin (N - \psi_0)$$
$$i \cos \Omega = I \cos (N - \psi_0) + \omega_0$$

If further we introduce the notations

$$\left.\begin{array}{ll} \Gamma_i = \psi_0 - \theta_i & \psi = \psi_0 - \theta_0 + 180^\circ \\ x_i = \gamma_i \sin \Gamma_i & x_0 = \omega \sin \psi \\ y_i = \gamma_i \cos \Gamma_i & y_0 = \omega \cos \psi - \omega_0 \end{array}\right\} \quad . \; . \; . \quad (2)$$

then the expressions for p and q become:

$$\left.\begin{array}{l} p_i = \sum_{j=1}^{4} \sigma_{ij} x_j - \mu_i x_0 \\ q_i = \sum_{j=1}^{4} \sigma_{ij} y_j + (1 - \mu_i) \omega_0 - \mu_i y_0 \end{array}\right\} \quad . \; . \; . \; . \quad (3)$$

MARTH has adopted

$$\left.\begin{array}{l} \omega_0 = \text{the value of } \omega \\ \psi_0 = \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad \theta_0 + 180^\circ \end{array}\right\} \text{ from SOUILLART's theory,}[1]$$

and has computed the values of p and q by the formulae (3), taking $x_0 = y_0 = 0$.

The unknowns γ_i, Γ_i, x_0 and y_0 must be determined from the equations (3). This is, of course, only possible if the coefficients σ_{ij} and μ_i are known. I have adopted these coefficients from SOUILLART's theory, as being the best available. They are very complicated functions of the masses, the compression of Jupiter, and the mean motions. As a rough approximation, we can say that the coefficients σ_{ij} are proportional to the mass m_j. Since the masses are very imperfectly known, the same thing is true of the coefficients of the equations (3). Therefore the results of the present discussion cannot be considered as final, but the discussion will have to be repeated when better values of the coefficients are available. The results here derived will however doubtlessly represent a very fair approximation.

It may perhaps be mentioned that the uncertainty of these coeffi-

[1]) MARTH has made one or two mistakes here, which will be duly mentioned in the detailed publication, but as they have no influence on the result they can be ignored at present.

cients is not due to our ignorance with respect to the masses alone. The values of these coefficients derived by SOUILLART from *the same* masses and elements by two different methods of integration show differences of such amount, that the consequent differences in the computed values of p and q are of the order of the errors of observation. It is hardly to be expected that this defect in the theory will be remedied before the equator is introduced instead of the orbit as the fundamental plane of the theory. The coefficients adopted by MARTH and myself are those derived from the second method of integration, which is also preferred by SOUILLART himself.

In the following discussions these coefficients are treated as absolute constants. If we denote the corrections to the adopted values of x_i and y_i by δx_i and δy_i, then the unknowns

$$\delta x_i \quad \delta y_i \quad x_0 \quad y_0$$

must be determined from the equations

$$\left.\begin{aligned} \Sigma\, \sigma_{ij}\, \delta x_j - \mu_i\, x_0 &= \Delta\, p_i \\ \Sigma\, \sigma_{ij}\, \delta y_j - \mu_i\, y_0 &= \Delta\, q_i \end{aligned}\right\} \quad \ldots \ldots \quad (4)$$

The term $(1 - \mu_i)\, \omega_0$ in the second equation (3) must, of course, be treated as rigorously known.

The solution of the equations (4) is conducted in the following manner. I define the quantities $\Delta\, x_i$ and $\Delta\, y_i$ by the equations

$$\left.\begin{aligned} \Sigma\, \sigma_{ij}\, \Delta\, x_j &= \Delta\, p_i \\ \Sigma\, \sigma_{ij}\, \Delta\, y_j &= \Delta\, q_i \end{aligned}\right\} \quad \ldots \ldots \quad (5)$$

These equations are solved once and for all, and the solution is:

$$\left.\begin{aligned} \Delta\, x_i &= \Sigma\, \sigma_{ij}'\, \Delta\, p_j \\ \Delta\, y_i &= \Sigma\, \sigma_{ij}'\, \Delta\, q_j \end{aligned}\right\} \quad \ldots \ldots \quad (6)$$

Further, if we put:

$$\mu_i' = \Sigma\, \sigma_{ij}'\, \mu_j$$

then the equations of condition become:

$$\left.\begin{aligned} \delta x_i - \mu_i'\, x_0 &= \Delta\, x_i \\ \delta y_i - \mu_i'\, y_0 &= \Delta\, y_i \end{aligned}\right\} \quad \ldots \ldots \quad (7)$$

Next, if we denote the originally adopted values of x_i and y_i by x_{i_0} and y_{i_0} so that $x_i = x_{i_0} + \delta x_i$, $y_i = y_{i_0} + \delta y_i$, then the equations become:

$$\left.\begin{aligned} x_i - \mu_i'\, x_0 &= x_{i_0} + \Delta x_i \\ y_i - \mu_i'\, y_0 &= y_{i_0} + \Delta y_i \end{aligned}\right\} \quad \ldots \ldots \quad (8)$$

In these equations x_i and y_i are defined by the equations (2), where the γ_i are constants, and $\Gamma_i = \Gamma_{i_0} + \frac{d\Gamma_i}{dt}(t - t_0)$. The unknowns, which must be determined from the solution of the equations (8) are x_0, y_0, γ_i, Γ_{i_0} and $\frac{d\Gamma_i}{dt}$.

The values of $\frac{d\Gamma_i}{dt}$ for the four satellites are however not mutually independent. The theory gives these differential coefficients as functions of the masses of the satellites and the compression of Jupiter. The masses need not be considered here. I have tried to determine a correction to m_3, but this determination had too small a weight to have any real value. The influence of the other masses is even smaller.

The compression enters into the formulas through the factor Jb^2, where J is the well known constant, which is approximately equal to $\varrho - {}^1/_2\,\varphi$ (ϱ = ellipticity of the free surface, φ = ratio of centrifugal force to gravity at the equator of Jupiter) and b is the equatorial radius [1]) of the planet.

If we introduce as unknown:

$$\varkappa = \frac{\delta Jb^2}{Jb^2}$$

then the true values of the coefficients of t are

$$\frac{d\Gamma_i}{dt} = \left(\frac{d\Gamma_i}{dt}\right)_0 + a_i\,\varkappa$$

The coefficients a_i depend practically alone on the mean motions, and must be treated as absolute constants. They differ little from $\frac{d\Gamma_i}{dt}$ itself, and consequently the *ratios* of the motions of the nodes must be considered as approximately constant. The adopted values according to SOUILLART's theory are (daily motions):

$$\left(\frac{d\Gamma_1}{dt}\right)_0 = 0^\circ.14109 \qquad \left(\frac{d\Gamma_3}{dt}\right)_0 = 0^\circ.007019$$

$$\left(\frac{d\Gamma_2}{dt}\right)_0 = 0^\circ.033010 \qquad \left(\frac{d\Gamma_4}{dt}\right)_0 = 0^\circ.001898$$

The 36 equations (8) thus contain the 11 unknowns

$$\gamma_i \quad \Gamma_{i_0} \quad x_0 \quad y_0 \quad \varkappa$$

These equations must be solved by successive approximations. The conditions for the application of the method of least squares are far from being fulfilled.

These approximations have been conducted in the following manner.

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[1]) In the original Dutch b was erroneously stated to be the diameter, instead of the radius.

Let x_{00} y_{00} be approximate values of x_0 and y_0, thus $x_0 = x_{00} + \delta x_0$ and $y_0 = y_{00} + \delta y_0$. We have then:

$$\left. \begin{aligned} x_i - \mu_i' \, \delta x_0 &= x_{i_0} + \Delta x_i + \mu_i' \, x_{00} \\ y_i - \mu_i' \, \delta y_0 &= y_{i_0} + \Delta y_i + \mu_i' \, y_{00} \end{aligned} \right\} \quad . \; . \; . \; . \; . \quad (9)$$

If we suppose that the approximation x_{00} y_{00} is already so good that δx_0 and δy_0 can be neglected, then these equations become:

$$\left. \begin{aligned} \gamma_i \sin \Gamma_i &= x_{i_0} + \Delta x_i + \mu_i' \, x_{00} \\ \gamma_i \cos \Gamma_i &= y_{i_0} + \Delta y_i + \mu_i' \, y_{00} \end{aligned} \right\} \quad . \; . \; . \; . \; . \; . \quad (10)$$

Next I compute the quantities g_i and G_i from the equations:

$$\left. \begin{aligned} g_i \sin G_i &= x_{i_0} + \Delta \, x_i + \mu_i' \, x_{00} \\ g_i \cos G_i &= y_{i_0} + \Delta \, y_i + \mu_i' \, y_{00} \end{aligned} \right\} \quad . \; . \; . \; . \; . \; , \quad (11)$$

The other unknowns are then determined from the equations:

$$\left. \begin{aligned} \gamma_i &= g_i \\ \Gamma_{i_0} + \frac{d\Gamma_i}{dt}(t - t_0) &= G_i \end{aligned} \right\} \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

If these equations give constant values for γ_i and values of $\frac{d\Gamma_i}{dt}$, which can by an acceptable value of $\varkappa$ be made consistent with the theory, then the appromation is sufficient, if not, then a new approximation must be made. As a first approximation I have assumed:

$$x_{00} = y_{00} = 0.$$

The equations (12) were then formed and solved. In this solution I have determined the values of $\frac{d\Gamma_i}{dt}$ for the four satellites separately without introducing the theoretical ratios ab initio. The equations (12) then consist of two sets for each satellite and each of these 8 sets is independent of all others. The residuals which remain after the substitution of the resulting values of the unknowns will be given below together with those from the other solutions. The probable error of unit weight was $\pm 0^\circ.0086$.

The motions of the nodes in this solution are (Sol. I):

$$\frac{d\Gamma_1}{dt} = 0^\circ.01213 \qquad \frac{d\Gamma_3}{dt} = 0^\circ.00587$$

$$\frac{d\Gamma_2}{dt} = 0^\circ.030266 \qquad \frac{d\Gamma_4}{dt} = 0^\circ.00189.$$

If these are compared with the theoretical values, it appears at once that their ratios are very different. The node of satellite I, which according to the theory has a yearly motion of about 50°, in this solution shows a motion of about 5°. The ratios of the three other motions also differ considerably from their theoretical values. Moreover the inclinations are far from constant, as will be seen at once from an inspection of the residuals $\Delta\gamma$.

It must be mentioned that the value of $\frac{d\Gamma_2}{dt}$ agrees approximately with the value derived by Cookson from the observations of 1891, 1901 and 1902. This could have been expected since Cookson in this determination also neglected the corrections to the position of the equator. The difference between Cookson's value of $\frac{d\Gamma_2}{dt}$ and the value of Sol. I is not due to a bad agreement of the observations of 1903 and 1904 with those of 1901 and 1902 (which on the contrary agree extremely well), but to the fact that in Sol. I the corrections to the elements of the other satellites were eliminated by means of the transformation from Δp and Δq to Δx and Δy, while Cookson did not eliminate these corrections but neglected them.

I have now made a number of further solutions, in which I started with approximate values x_{00} and y_{00}, and introduced the unknowns

$$\gamma_i \qquad \Gamma_{i_0} \qquad \delta x_0 \qquad \delta y_0 \qquad \varkappa,$$

thus rigorously subjecting the motions of the nodes to the theoretical condition. The unknowns δy_0 and $\varkappa$ are badly separated. The weight of the determination of $\varkappa$ is considerably diminished by the introduction as unknowns of the corrections to the position of the equator. That this must of necessity be so, is easily seen. If we had observations of only one satellite at two epochs, it would be *impossible* to determine both the motion of the node and the equator. We would in that case have only four data (the values of p and q at each of the epochs) for the determination of the five unknowns γ, Γ, $\frac{d\Gamma}{dt}$, x_0, and y_0. Now $\varkappa$ is practically determined from satellite II alone. The motions of the nodes of III and IV are too slow, and the inclination of I is too small, to allow a determination of the motions of the nodes of these satellites to be made, the accuracy of which would be even remotely comparable to that of sat. II. The motions of the nodes of I, III and IV are derived theoretically from that of II. If therefore the latter is known, each of the three others provides a determination of the equator. Then the determination of $\varkappa$ from II must be repeated with this new position of the equator, and so on until a satisfactory agreement is reached. [1])

[1]) Cookson has in his discussion of the observations of 1891, 1901 and 1902, used this method, but he rested content with the first approximation. His corrections to the equator derived from satellites III and IV are in the same direction as the values found by me.

The solution was not actually made in this way, but all equations were treated simultaneously. This consideration is only given here to point out that the position of the equator is ultimately determined by the condition that it shall be the same for the four satellites, i. e. that the inclinations shall be constant, and the motions of the nodes shall be consistent with the theoretical ratios. Since a small displacement of the equator has a large influence on the motions of the nodes, in consequence of the small inclinations, it can be expected that the unknown $\varkappa$ and the quantities which determine the position of the equator will mutually diminish each others weights. (That this decrease of weight is actually much more marked in the case of y_0 than for x_0, is accidental and depends on the choice of the zero of longitudes).

By these considerations I have been led to try whether the value of $\varkappa$ could not be determined from a comparison with other observations. I have used the values of θ_i for 1750 given by DELAMBRE. A value of $\varkappa$ was adopted, such that the value of θ_2 carried back to 1750 from the modern observations would be nearly equal to the value given by DELAMBRE. The unknowns x_0, y_0, $\delta\gamma_i$ and $\delta\Gamma_{i0}$ were then determined from the modern observations alone.

This gives solution VII. In solution VI on the other hand all unknowns (inclusive of $\varkappa$) were determined from the modern observations. I give below the results from these two solutions, which I consider as the best that can be derived with our present knowledge of the masses. I do not venture to choose between the two solutions. Probably an eventual correction of the coefficients σ_{ij} will tend to reconcile the two solutions.

Instead of Γ_i I give at once $\theta_i = \psi_0 - \Gamma_i$. The values are given for 1900 Jan. 0 Greenwich Mean Noon.

	Solution VI	*Solution VII*	*Adopted values.*
x_0	$-\ 0^\circ.0172 \pm {}^\circ.0023$	$-0^\circ.0177 \pm .^\circ0022$	0
y_0	$+\ 0\ .0427 \pm\ .0043$	$+0\ .0489 \pm .\ 0022$	0
$\varkappa$	$-\ 0.0321 \pm\ .0094$	$-\ 0.0126$	0
γ_1	$0^\circ.0259 \pm {}^\circ.0032$	$0^\circ.0248 \pm\ .0038$	$0^\circ.0013$
γ_2	$.4696 \pm\ 27$	$.4676 \pm\ 24$	.4694
γ_3	$.1926 \pm\ 40$	$.1874 \pm\ 26$	.1789
γ_4	$.2540 \pm\ 34$	$.2504 \pm\ 25$	.2254
θ_1	$54^\circ.4\ \pm 8^\circ.5$	$54^\circ.0\ \pm 8^\circ8$	$99^\circ.8$
θ_2	$293\ .42 \pm 0\ .35$	$293\ .10 \pm 0.29$	273 .32
θ_3	$319\ .68 \pm 0\ .77.$	$319\ .67 \pm 0.80$	330 .59
θ_4	$14\ .40 \pm 0\ .91$	$15\ .56 \pm 0,57$	5 .79

From the values of $\varkappa$ we find the following values of $\frac{d\theta}{dt}$

$\frac{d\theta_1}{dt}$	$-$ 0°.13664	$-$ 0°.13932	$-$ 0°.14105
$\frac{d\theta_2}{dt}$	$-$ 0 .032105	$-$ 0 .032633	$-$ 0 .032974
$\frac{d\theta_3}{dt}$	$-$ 0 .006814	$-$ 0. 006916	$-$ 0 .006983
$\frac{d\theta_4}{dt}$	$-$ 0 .001839	$-$ 0 .001854	$-$ 0 .001863

From the values of x_0 and y_0 we find for the inclination and node of the equator on LEVERRIER's orbit of Jupiter of 1900·0:

ω	3°.1107 ± °·0043	3°·1169 ± °.0022	3°.0680
θ	315.727 ± ·042	315.735 ± ·041	315.410

With the exception of $\varkappa$ all unknowns in the two solutions agree within the sum of their probable errors, and with only one exception (γ_2) all the corrections to the adopted values are many times larger than their probable errors.

The residuals of the two solutions VI and VII are given in the following table together with those of Sol. I. The probable errors, which have been added for comparison are somewhat larger than those of the observed Δp and Δq, because by the transformation from Δp and Δq to Δx and Δy, the p.e. must be somewhat increased, even if we consider the coefficients σ_{ij} as absolutely exact.

The p.e. of weight unity, which was ± 0°.0086 for Sol. I, is ± 0°.0065 for Sol. VI and ± 0°.0064 for Sol. VII. But it is chiefly in their consistency with the theoretical conditions, that both solutions are incomparably better than Sol. I. The inclinations are now constant within the probable errors. The residuals of the nodes only show a systematic tendency for Satellite I (in Sol. VII, where the motions of the nodes were not derived from the observations, also for Sat. II). Still the agreement with the theoretical motions is much improved. The value of $\frac{d\Gamma_1}{dt}$ derived from Sol. VI irrespective of the theoretical conditions would be 0°.1250, while the value corresponding to the value of $\varkappa$ in this solution is 0°.1366. This is a great improvement compared with Sol. I (0°.0121).

The results for Sat. III in 1901 and 1902, which in all solutions gave large residuals, have in the solutions VI and VII been rejected. This rejection has no appreciable influence on the values of the unknowns, nor on the other residuals, but it reduces the p. e. of

		p. e.	Sol. I		Sol. VI		Sol. VII	
			Δγ	sin γΔΓ	Δγ	sin γΔΓ	Δγ	sin γΔΓ
Sat. I.	1891	±°.0045	−°.0068	−°.0003	+°.0005	+°.0123	+°.0047	+°.0093
	1901	± 85	+ 157	− 32	− 12	+ 101	− 60	+ 99
	02	± 75	+ 213	− 38	+ 55	− 97	+ 8	− 100
	03	± 60	+ 37	+ 34	− 33	− 155	− 61	− 112
	04	± 80	− 237	+ 48	− 5	− 99	+ 38	− 78
Sat. II.	1891	±.0030	+.0138	+.0002	+.0017	−.0008	+.0016	+.0045
	1901	± 60	+ 137	− 40	+ 73	+ 20	+ 56	+ 8
	02	± 50	− 73	+ 2	− 63	+ 50	− 65	+ 29
	03	± 40	− 95	+ 6	− 2	+ 4	+ 1	− 29
	04	± 50	− 210	+ 8	− 30	− 54	− 13	− 104
Sat. III.	1891	±.0020	+.0048	+.0007	−.0014	−.0029	−.0013	−.0037
	1901	± 40	− 137	− 33	[− 178]	[− 29]	[− 181]	[− 24]
	02	± 30	− 114	− 77	[− 152]	[− 73]	[− 155]	[− 67]
	03	± 25	+ 32	+ 57	+ 11	+ 56	+ 10	+ 62
	04	± 30	+ 33	− 30	+ 14	− 31	+ 12	− 22
Sat. IV.	1891	±.0010	−.0010	+.0001	+.0013	−.0028	+.0017	−.0031
	1901	± 20	[− 51]	[+ 188]	[− 101]	[+ 205]	[− 110]	[+ 200]
	02	± 20	[− 86]	[− 185]	[− 83]	[− 166]	[− 85]	[− 171]
	03	± 15	+ 22	+ 28	− 4	+ 61	− 7	+ 62
	04	± 20	− 11	− 60	− 30	− 20	− 34	− 21

weight unity from ± 0°.0072 and ± 0°.0073 to ± 0°.0065 and ± 0°.0064 for the solutions VI and VII respectively.

The values of θ_i carried back to 1750 are:

	Sol. I	Sol. VI	Sol. VII	Damoiseau	Delambre
θ_2	151°.8	252°.4	281°.0	282°.0	283°.3
θ_3	282 .9	333 .0	338 .6	353 .5	352 .5
θ_4	110 .3	114 .2	117 .1	98 .3	105 .0

In conclusion I must express my deep sense of gratitude towards Sir DAVID GILL, who liberally placed the observations of the Cape Observatory at my disposal, and was always ready to meet all my wishes.

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