Citation:

F.M.Jaeger, A simple geometrical deduction of the relations existing between known and unknown quantities, mentioned in the method of Voigt for determining the conductibility of heat in crystals, in: KNAW, Proceedings, 8 II, 1905-1906, Amsterdam, 1906, pp. 793-797

Nous en connaissons de nombreux exemples[1]). If the stated phenomena are connected with each other the conformity to the earlier process of development of the set of teeth of the Primates as I take it, immediately strikes the eye; and one would be inclined to this thesis; In the future set of teeth of man P_2 will no longer erupt, m_2 will have become persistent and functionate as M_1, but by the simultaneous reduction of M_3 the number of molars will not have become larger than three.

So from this communication appears that the differentiation of the entire set of teeth of the Primates is from my standpoint more intricate than was supposed till now, but it seems to me that my principle of the terminal reduction can better be brought into accordance with the function of the set of teeth, and is based on a larger number of facts than the hypothesis of the excalation. What from a general point of view, also seems to me to plead for my opinion is the fact, that in the exposition given by me the development of the set of teeth has taken place without a discontinuity in the toothrows at any time.

Physics. — *"A simple geometrical deduction of the relations existing between known and unknown quantities, mentioned in the method of* VOIGT *for determining the conductibility of heat in crystals.* By Dr. F. M. JAEGER. (Communicated by Prof. P. ZEEMAN.)

(Communicated in the meeting of March 31, 1906).

It is commonly known that about ten years ago W. VOIGT[2]) indicated a method, based on a recognized principle of KIRCHHOFF, by which to determine the relative conductibility of heat in crystals in the different directions. His mode of experimental examination consists in the determination of the break which two isothermal lines present at the boundary line of an artificial twin, the principal directions of which form a given angle φ with that line, whilst the conduction of heat takes place along the line of limit. The isothermal lines are rendered visible to the eye by the tracings formed by the fusion of a mixture of elaidic acid and wax with which the plane of the crystal has previously been covered.

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1) E. MAGITOT. Traité des Anomalies du Système dentaire. Paris 1877. p. 221.

2) VOIGT, Göttinger Nachrichten, 1896, Heft 3.

The method of VOIGT is far more accurate than that of DE SÉNARMONT[1]) or even of RÖNTGEN[2]), and, requiring for other purposes to investigate the relative conductibility of heat in crystals, it was obvious that I should make use of the method indicated by VOIGT.

For a crystal, for which the rotatory coefficients, found in accordance with the theory of G. C. STOKES[3]), are $= 0$, VOIGT deducts the relations required here by constructing the equations of the flow of heat, conformable to the conditions of limit which are common to the lateral boundaries of both plates; i.e. that *along* that line the loss of temperature must be the same, and moreover that in a direction normal to that boundary-line the entire flow of heat must be the same in the two contiguous plates.

Prof. LORENTZ had the kindness to derive the above mentioned relations in an analogous manner and to note down the conditions under which the break in the isothermal lines will reach its maximum.

If ε be the break, and ϕ the angle, formed in the plates by the two principal directions, is 45°, the proportion of the two coefficients of the conduction of heat in those directions, consequently $\frac{\lambda_1}{\lambda_2}$ is found as follows:

$$tg\varepsilon = (\lambda_1 - \lambda_2)\frac{(\lambda_1 + \lambda_2)}{2\lambda_1\lambda_2}.$$

If φ differs from 45°, VOIGT finds in that case:

$$tg\beta = \frac{(\lambda_1 - \lambda_2)\sin 2\varphi}{(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)\cos 2\varphi},$$

which for φ equal to 45° passes into the formula of Prof. LORENTZ by introducing $tg\frac{\varepsilon}{2}$ ($= tg\beta$ according to VOIGT's deduction) instead of $tg\varepsilon$.

Instead of the complicated formulae which are required for the determination of these relations, we here give a simple geometrical demonstration, which, besides presenting $\frac{\lambda_1}{\lambda_2}$ in a form which is immediately available for logarithmic calculations, possesses at the same time the advantage of being easily discernible.

If, from a given point O in the centrum of a crystal, a flow of heat can take place without interruption in all directions, the isothermal

1) DE SÉNARMONT, Compt. rend. **25**, 459, 707. (1847).

2) RÖNTGEN, Pogg. Ann. **151**, 603, (1874).

3) STOKES, Cambr. and Dublin Math. Journal. **6** 215, (1851).

surfaces in a similar plane of a crystal are, in most cases, concentric and equiform three-axial ellipsoids whose half axes stand in the relation of $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$; among these the so-called principal ellipsoid h, whose axes are $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$ must here be kept more especially in view.

In the present case we leave unnoticed the rotatory qualities of the crystal, and suppose an infinitely thin plate, cut parellel to a plane of thermic symmetry, whose principal directions correspond to the coordinate axes. Let fig. 1 represent the elliptic intersection of the plate with the ellipsoid h; the line traced by the melted wax then has the direction of the tangent of the ellipse in the point $P(x'y')$, given by the radius vector ϱ, which may enclose the angle φ with the axis X. The flow of heat may thus proceed along ϱ, being the boundary line. In this case the equation for the isothermal line pq is:

$$\frac{xx'}{\lambda_1} + \frac{yy'}{\lambda_2} = 1.$$

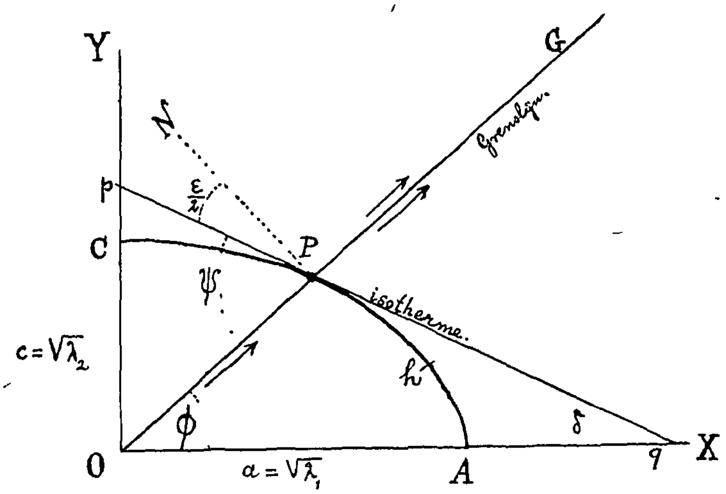

Fig. 1.

Thus for the two sections Op and Oq cut off on the two axes the result is:

$$O_p = \frac{\lambda_2}{y'} = \frac{\lambda_2}{\varrho \sin \varphi}$$

$$O_q = \frac{\lambda_2}{x'} = \frac{\lambda_1}{\varrho \cos \varphi}$$

therefore:

$$\frac{O_p}{O_q} = \frac{\lambda_2}{\lambda_1} \cot \varphi.$$

On the other hand however:

$$\frac{O_p}{O_q} = tg\,\delta = tg\left\{90^\circ - \left(\varphi + \frac{\varepsilon}{2}\right)\right\} = cot\left(\varphi + \frac{\varepsilon}{2}\right),$$

where $\frac{\varepsilon}{2}$ is half of the break of the isothermal lines at the boundary line OG.

The immediate conclusion is therefore:

$$\frac{\lambda_1}{\lambda_2} = tg\left(\varphi + \frac{\varepsilon}{2}\right) cot\,\varphi \quad \ldots\ldots\ldots \quad (A)$$

From this equation the required proportion may be at once deduced when φ represents the direction of the plate and the value of ε has been ascertained.

Moreover it will be easy to find the maximum of ε — and thus reduce the errors of investigation to the lowest figures. Suppose $A = \frac{\lambda_1}{\lambda_2}$, the above stated formula, after a few goniometrical transformations becomes:

$$tg\,\frac{\varepsilon}{2} = \frac{(A-1).\,sin\,2\varphi}{(A+1) - (A-1)\,cos\,2\varphi}.$$

This function will be a maximum for $\frac{d\varepsilon}{d\varphi} = 0$, i. e.

$$\frac{d\varepsilon}{d\varphi} = \frac{2\left\{(A^2-1)\,cos\,2\varphi - (A-1)^2\right\}}{(A^2+1) - (A^2-1)\,cos\,2\varphi} = 0.$$

The maximum condition then becomes:

$$cos\;2\varphi = \frac{A-1}{A+1} = \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2},$$

and the appertaining maximum break ε in the isothermal lines is then expressed by:

$$tg\,\frac{\varepsilon}{2} = \frac{(\lambda_1-\lambda_2)}{2\sqrt{\lambda_1\lambda_2}} \quad \ldots\ldots\ldots \quad (B)$$

In cases where the difference between $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ is very small — and observation teaches that this is usually the case — the notation may be:

$$tg\,\frac{\varepsilon}{2} = \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2} \quad \ldots\ldots\ldots \quad (C)$$

For practical purposes therefore, the theoretical maximum $\varphi = 45^\circ$ may be taken as fairly accurate, so that then the twin plate with the isothermal lines etc., takes the form of fig. 2. In that case it follows from A:

$$\frac{\lambda_1}{\lambda_2} = tg\left(45^\circ + \frac{\varepsilon}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D)$$

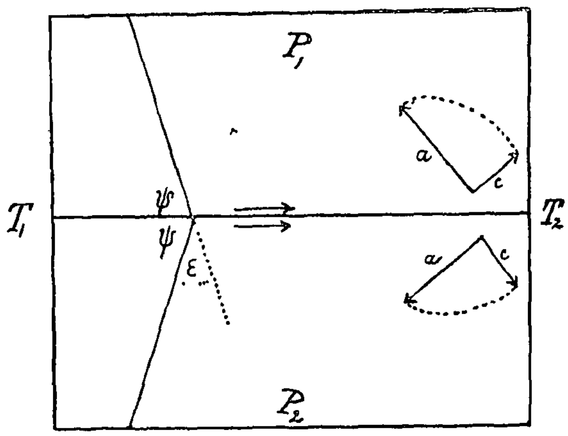

Fig. 2.

By expressing $tg\, \varepsilon$ as a function of $tg\, \frac{\varepsilon}{2}$ from (C) one obtains the relation deduced by Prof. LORENTZ;

$$tg\, \varepsilon = (\lambda_1 - \lambda_2)\, \frac{(\lambda_1 + \lambda_2)}{2\, \lambda_1\, \lambda_2}.$$

Moreover from the geometrical solution here given the fact is again brought to light that in general the angle ψ is not equal to 90°; in other words in this simple but experimental way is proved by occular demonstration the truth of the statement already made by VOIGT, that the isothermal lines in crystals do not generally stand perpendicular to the direction of the flow of heat.

Along the thermic axes however this is the case, because the tangent lines at the ellipses are there directed perpendicularly to these axes.

From fig. 1 also follows the form of the break as a result of $\lambda_1 \gtrless \lambda_2$.

I hope soon to communicate the results obtained in the measurement of crystals by means of this method, together with a few observations on the differences of these results with those, derived in the same minerals by the usual methods of DE SÉNARMONT and RÖNTGEN.