Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW)

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The magnetisation of the spectral lines enables us to determine the maximum value of the force with phenomena varying rapidly with the time, and with non-uniform fields.

In some cases it is of great importance to follow the behaviour of a spectral phenomenon with different strengths of field. The above described method might then be called *the method of the non-uniform field.*

In a future communication I hope to study in this manner the *asymmetry* of the separation of spectral lines in weak magnetic fields, predicted from theory by VOIGT. On a former occasion I have communicated some experiments giving rather convincing evidence of the existence of this asymmetry [1]).

In the mean time, I think that the developments lately given by LORENTZ [2]) make it desirable to corroborate the reasons for accepting the existence of this extremely small asymmetry.

Mathematics. — *"Some properties of pencils of algebraic curves".* By Prof. JAN DE VRIES.

§ 1. Let A be one of the n^2 basepoints of a pencil (c^n) of curves c^n of order n, B one of the remaining basepoints. If we make to correspond to each c^n the right line c^1 touching c^n in A, then we get as product of the projective pencils (c^n) and (c^1), a curve T_1 of order $(n+1)$ forming the locus of the tangential points of A, i. e. of the points which are determined by each c^n on its tangent c^1. This *tangential curve* has in A a threefold point where it is touched by the inflectional tangents of three c^n having in A an inflection; it has been considered for the first time by EMIL WEYR (Sitz. Ber. Akad. in Wien, LXI, 82).

I shall now consider more in general the locus T_m of the m^{th} tangential points of A. The order of this curve is to be represented by $\tau(m)$, whilst $\alpha(m)$ and $\beta(m)$ are to indicate the number of branches which T_m has in A and B.

Prof. P. H. SCHOUTE has drawn my attention to a paper inserted by him in the Comptes Rendus de l'Académie des sciences, tome CI, 736, where the corresponding curve is treated for a cubic pencil. I found that the numbers obtained there for $n = 3$ appear from the results to be deduced here.

1) ZEEMAN. These Proceedings, December 1899.

2) LORENTZ. These Proceedings, December 1905.

To determine the functions $\tau(m)$, $\alpha(m)$ and $\beta(m)$ I shall make use of an auxiliary curve already used by WEYR, which might be called the *antitangential curve* of A. It contains the groups of $n(n-1)-2$ points A_{-1}, having A as tangential point; so it passes three times through A and once through all points B. So it has $(2n^2-n)$ points in common with any c^n, from which it is evident that it is of order $(2n-1)$.

§ 2. The $(m-1)^{\text{th}}$ tangential curve (A^{m-1}) of A is cut by the antitangential curve (A^{-1}) of A, save in the base points, in the points $A^{(m-1)}$ having A as tangential point. Their number amounts to three less than the number of tangents which T_m has in A, so $\alpha(m)-3$; for, on the three c^n which have in A an inflection A coincides with one of its m^{th} tangential points.

The three inflectional tangents being also tangents of the curve (A^{-1}), the tangential curve (A^{m-1}) and the antitangential curve (A^{-1}) have $3\alpha(m-1)+3$ points in common in A. In each base-point B lie $\beta(m-1)$ points of intersection. So

$$(2n-1)\,\tau(m-1) = \alpha(m) + 3\alpha(m-1) + (n^2-1)\,\beta(m-1) \,.\,. \quad (1)$$

A second relation is found by noticing that (A^{m-1}) has with the antitangential curve of B, save the basis, the $\beta(m)$ points in common for which B is an m^{th} tangential point. In B lie $3\beta(m-1)$ points of intersection, $\alpha(m-1)$ points of intersection lie in A, $\beta(m-1)$ in each of the other basepoints. So

$$(2n-1)\,\tau(m-1) = \beta(m) + \alpha(m-1) + (n^2+1)\,\beta(m-1) \quad . \quad (2)$$

With any c^n the locus T_m has, save the basis, only the $(n-2)^m$ points $A^{(m)}$ in common; so

$$n\,\tau(m) = \alpha(m) + (n^2-1)\,\beta(m) + (n-2)^m \quad . \; . \; . \quad (3)$$

§ 3. To find a homogeneous equation of finite differences for the determination of $\tau(m)$ I eliminate from the three obtained relations the quantities $\alpha(m)$ and $\beta(m)$, and I find

$$n\tau(m) = n^2(2n-1)\tau(m-1) - (n^2+2)\{\alpha(m-1)+(n^2-1)\beta(m-1)\} + (n-2)^m.$$

Here the expression within braces can be replaced on account of (3) by $n\tau(m-1)-(n-2)^{m-1}$. Then

$$\tau(m) = (n^2-n-2)\,\tau(m-1) + (n+1)(n-2)^{m-1} \quad . \; . \quad (4)$$

So

$$\tau(m-1) = (n^2-n-2)\,\tau(m-2) + (n+1)(n-2)^{m-2} \quad . \quad (5)$$

Equations (4) and (5) finally furnish

$$\tau(m) - (n-2)(n+2)\,\tau(m-1) + (n-2)^2(n+1)\,\tau(m-2) = 0 \quad (6)$$

To determine a particular solution $\tau(m) = x^m$ we have

$$x^2 - (n-2)(n+2)x + (n-2)^2(n+1) = 0,$$

therefore

$$x = n^2 - n - 2 \text{ or } x = n - 2.$$

Consequently the general solution is

$$\tau(m) = c_1(n^2 - n - 2)^m + c_2(n-2)^m.$$

To determine the constants c_1 and c_2 I substitute in (4) the known values $(n+1)$ of $\tau(1)$ and $(n+1)(n^2-4)$ of $\tau(2)$.

Now

$$n + 1 = c_1(n^2 - n - 2) + c_2(n-2),$$

$$(n^2-4)(n+1) = c_1(n^2 - n - 2)^2 + c_2(n-2)^2.$$

Finally we find by elimination of c_1 and c_2

$$\tau(m) = (n+1)(n-2)^{m-1} \frac{(n+1)^m - 1}{n} \quad . \quad . \quad . \quad . \quad (7)$$

From (1) and (2) ensues

$$\alpha(m) - \beta(m) = -2\{\alpha(m-1) - \beta(m-1)\},$$

so

$$\alpha(m) - \beta(m) = (-2)^{m-1}\{\alpha(1) - \beta(1)\} = -(-2)^m \quad . \quad . \quad (8)$$

Making use of (3) and (7) we now find

$$n^2\alpha(m) = (n-2)^{m-1}\{(n+1)^{m+1} - 2n + 1\} - (n^2-1)(-2)^m \quad . \quad (9)$$

$$n^2\beta(m) = (n-2)^{m-1}\{(n+1)^{m+1} - 2n + 1\} + (-2)^m \quad . \quad . \quad . \quad . \quad (10)$$

§ 4. For $m = 2$ we find $\alpha(2) = n^2 + n - 9$; as A is inflection for three curves c_n, there are therefore $(n^2 + n - 12)$ curves on which A coincides with its second tangential point. From this ensues the wellknown result that A is point of contact of $(n+4)(n-3)$ double tangents.

In a former paper [1]) I have brought into connection the locus of the points of contact D of the double tangents with the locus of the points W in which a c^n is cut by its double tangents. To determine, how often a point D coincides with one of its tangential points W I consider the correspondence of the rays $d = OD$ and $w = OW$ which the correspondence (D, W) forms in a pencil with vertex O.

As the curves (D) and (W) are of orders $(n-3)(2n^2 + 5n - 6)$ and $\frac{1}{2}(n-4)(n-3)(5n^2+5n-6)$, to each ray d correspond $(n-4)(n-3)(2n^2+5n-6)$ rays w and to each ray w correspond $(n-4)(n-3)(5n^2+5n-6)$ rays d.

1) "On linear systems of algebraic plane curves." Proc. April 22 1905, Vol. VII (a), p. 711.

Because each of the $2n(n-2)(n-3)$ double tangents out of O represents $2(n-4)$ coincidences $d \equiv w$, the number of coincidences $D \equiv W$ is represented by

$$(n-4)(n-3)(2n^2+5n-6)+(n-4)(n-3)(5n^2+5n-6)-$$
$$-4(n-4)(n-3)(n-2)n = 3(n-4)(n-3)(n^2+6n-4).$$

In a pencil (c^n) *we find that* $3(n-4)(n-3)(n^2+6n-4)$ *curves have an inflection, of which the tangent touches the curve in one other point more.*

In the paper quoted above I thought I was able to determine this number out of the points of intersection of the curves (D) and (W); here I overlooked the fact that a point of contact of a double tangent can be tangential point W of another double tangent.

§ 5. To find the number of threefold tangents I consider the correspondence between the rays projecting out of O two points W and W' lying on the same double tangent. The characterizing number of this symmetric correspondence is evidently equal to $\frac{1}{2}(n-4)(n-3)(5n^2+5n-6)(n-5)$, whilst each double tangent borne by O replaces $2n(n-2)(n-3)(n-4)(n-5)$ coincidences. The number of coincidences $W \equiv W'$ amounts thus to

$$(n-5)(n-4)(n-3)(5n^2+5n-6-2n^2+4n).$$

As each threefold tangent bears three of these coincidences we have the property:

In a pencil (c^n) *we find that* $(n-5)(n-4)(n-3)(n^2+3n-2)$ *curves have a threefold tangent.*

§ 6. In my paper indicated above I have tried to determine the number of undulation-points out of the points of intersection of the inflectional curve (I) with the locus of the points (V) which c^n determines on its inflectional tangents. As each inflection which is also tangential point of another inflection is common to (I) and (V), the number found elsewhere is too large. The exact number I can determine by means of the correspondence between the rays OI and OV.

As the orders of (I) and (V) are $6(n-1)$ and $3(n-3)(n^2+2n-2)$ and each of the $3n(n-2)$ inflectional tangents drawn from O replaces $(n-3)$ rays of coincidence, we get for the number of coincidences $I \equiv V$

$$6(n-1)(n-3)+3(n-3)(n^2+2n-2)-3n(n-2)(n-3) = 6(n-3)(3n-2).$$

In a pencil (c^n) *we find that* $6(n-3)(3n-2)$ *curves have a four-point tangent.*

§ 7. The curve of inflections (I) and the bitangential curve (D) have in each of the $3(n-1)^2$ nodes of (c^n) in common a number of $2(n-3)(n+2)$ points.

For, out of a node we can draw to the c^n to which it belongs (n^2-n-6) tangents, to be regarded as double tangents, whilst each node of a c^n is at the same time node of (I).

In each basepoint lie moreover $3(n+4)(n-3)$ points of intersection (§ 4). The remaining points common to (D) and (I) are the inflections of which the tangent touches the c^n once more (§ 4) and the undulation-points (§ 6) where the two curves touch each other.

Indeed, we have

$$6(n-1)^2(n-3)(n+2)+3n^2(n+4)(n-3)+3(n-4)(n-3)(n^2+6n-4)+$$
$$+12(n-3)(3n-2)=6(n-1)^2(n-3)(n+2)+3(n-3)(2n^3+6n^2-16n+8)=$$
$$=6(n-1)(n-3)(2n^2+5n-6),$$

and this is the product of the orders of (I) and (D).

Physiology. *"On the strength of reflex stimuli as weak as possible."* By Prof. H. ZWAARDEMAKER. (Report of a research made by D. I. A. VAN REEKUM).

(Communicated in the meeting of March 31, 1906).

Investigated were chemical, thermal, mechanical and electrical stimuli, which partly acted upon the skin partly on the sensible nerves of the animals, which were experimented on.

§ 1. The chemical stimuli were applied by immerging the hindleg of a winterfrog in a little bowl with a solution of sulphuric acid varying from $^1/_4$ to $^1/_{32}$% $\left(\frac{n}{20} \text{ to } \frac{n}{160}\right)$. The spinal cord system was withdrawn in the usual way from the influence of the cerebrum. After the experiment the legs were washed with distilled water and the experiment repeated after a pause of 5 minutes. Neglecting the preliminary reflex, only a complete reflex was considered as a positive result. After-reflexes and general movements did only show themselves when rather strong concentrations were used.

As a rule a $^1/_{17}$% $\left(\frac{n}{85}\right)$ solution of sulphuric acid may be accepted as the minimum stimulus which still produces reflexes. The reflex-